MA 214 002 Calculus IV (Spring 2106)

Exam 1 Solutions

1. (a) (18%) Find all solutions of the differential equation

$$\frac{dy}{dt} = \frac{y^2}{t-2}.$$

(b) (7%) Find the solution of the given equation that satisfies the initial condition $y(1) = -1$. Determine the largest interval of t on which this solution is defined.

Solution: (a) The given equation is separable. For $y \neq 0$, the equation can be recast as

$$\frac{dy}{y^2} = \frac{dt}{t-2},$$

from which we obtain

$$\int \frac{dy}{y^2} = \int \frac{dt}{t-2} + C, \qquad \text{or} \qquad -\frac{1}{y} = \ln|t-2| + C.$$

Solving the last equation for y, we get the solution

$$y = -\frac{1}{\ln|t-2| + C}. \tag{1}$$

Clearly $y = 0$ is also a solution of the given equation.

(b) Using the initial condition $y(1) = -1$, we obtain from (1) that

$$-1 = -\frac{1}{\ln|1-2| + C} = -\frac{1}{C},$$

which implies $C = 1$. Hence the solution that satisfies the initial condition $y(1) = -1$ is

$$y = -\frac{1}{\ln|t-2| + 1}. \tag{2}$$

The expression on the right-hand side of (2) is undefined at $t = 2$ and at the values of t where $1 + \ln|t-2| = 0$, i.e., $t = 2 - e^{-1}, 2 + e^{-1}$. These singular points divide $(-\infty, \infty)$ into four intervals, namely $(-\infty, 2 - e^{-1}), (2 - e^{-1}, 2), (2, 2 + e^{-1}), (2 + e^{-1}, \infty)$. Since $t_0 = 1 \in (-\infty, 2 - e^{-1})$, the interval of definition of solution (2) is $(-\infty, 2 - e^{-1})$.

2. A retiree has a sum S invested so as to draw interest at a fixed annual rate r compounded continuously. Withdrawals for living expenses are made at a constant rate of k dollars/year; assume that the withdrawals are made continuously.

   (a) (8%) If the initial value of the investment is S_0, formulate the initial-value problem that governs $S(t)$, the value of the investment at any time t.

   (b) (4%) Determine the equilibrium solution of the differential equation that governs $S(t)$.

   (c) (10%) Solve the initial-value problem you formulated to determine $S(t)$.

   (d) (3%) For a given S_0 and r, determine the withdrawal rate k_0 at which $S(t)$ will remain unchanged (i.e., $S(t) = S_0$ for all $t \geq 0$).

Solution: (a) The initial-value problem in question is:

$$\frac{dS}{dt} = rS - k, \qquad S(0) = S_0. \tag{3}$$

(b) The equilibrium solution is given by the equation $rS - k = 0$ or $S = k/r$.

(c) Equation $(3)_1$ is linear. An integrating factor of the equation is $\mu = e^{-rt}$. Multiplying both sides of the equation $S' - rS = -k$ by μ, we obtain

$$\frac{d}{dt}\left(e^{-rt}S\right) = -k\, e^{-rt},$$

which implies after integration

$$e^{-rt}S = \frac{k}{r}e^{-rt} + C, \qquad \text{or} \qquad S(t) = \frac{k}{r} + C\, e^{rt}.$$

Substituting $t = 0$ into the preceding equation and using the initial condition $S(0) = S_0$, we get

$$C = S_0 - \frac{k}{r}.$$

Hence the solution of the initial-value problem for S is

$$S(t) = \frac{k}{r} + \left(S_0 - \frac{k}{r}\right) e^{rt}.$$

(d) For the equilibrium solution $S(t) = S_0$, we have $k_0 = r\, S_0$.

3. (a) (10%) Show that the differential equation

$$\left(y^2+3x\right)dx+\left(xy-\frac{3y^2}{x}\right)dy=0$$

is not exact, but $\mu(x)=x$ is an integrating factor of the equation.

(b) (15%) Use the integrating factor to solve the given equation.

Solution: Let $M=y^2+3x$, and $N=xy-3y^2/x$. Then

$$\frac{\partial M}{\partial y}=2y, \qquad \frac{\partial N}{\partial x}=y+\frac{3y^2}{x^2}.$$

Since $\partial M/\partial y \neq \partial N/\partial x$, the given equation is not exact.

After multiplying the equation by $\mu=x$, the equation becomes

$$\left(xy^2+3x^2\right)dx+(x^2y-3y^2)dy=0.$$

Let $\widetilde{M}=xy^2+3x^2$, and $\widetilde{N}=x^2y-3y^2$. Since

$$\frac{\partial \widetilde{M}}{\partial y}=2xy=\frac{\partial \widetilde{N}}{\partial x},$$

the equation has become exact.

We seek a function $\psi=\psi(x,y)$ such that $\partial\psi/\partial x=\widetilde{M}$ and $\partial\psi/\partial y=\widetilde{N}$. Integrating both sides of the equation $\partial\psi/\partial x=xy^2+3x^2$ with respect to x while keeping y fixed, we obtain

$$\psi(x,y)=\frac{x^2y^2}{2}+x^3+f(y), \tag{4}$$

where f is a differentiable function of y. Taking the partial derivative of both sides of (4) with respect to y, we have

$$\frac{\partial\psi}{\partial y}=x^2y+\frac{df}{dy}.$$

Since $\partial\psi/\partial y=\widetilde{N}=x^2y-3y^2$, we conclude that $df/dy=-3y^2$. Without loss of generality, we take $f(y)=-y^3$ and $\psi(x,y)=\frac{1}{2}x^2y^2+x^3-y^3$. The solutions of the given differential equation are given implicitly by the equation

$$\frac{1}{2}x^2y^2+x^3-y^3=C,$$

where C is a constant.

4. (a) (10%) Find the roots of the characteristic equation of the differential equation

$$4y'' + y' = 0,$$

and use them to write down the general solution of the differential equation.

(b) (15%) Solve the initial-value problem

$$y'' - y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Solution: (a) The characteristic equation in question is $4r^2 + r = r(4r+1) = 0$, the roots of which are: $r_1 = 0$, $r_2 = -1/4$. The general solution of the given differential equation is:

$$y = c_1 + c_2 e^{-t/4},$$

where c_1 and c_2 are constants.

(b) Here the characteristic equation is: $r'' - r + 1 = 0$. By the quadratic formula, the roots are found to be

$$r = \frac{2 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

Hence the general solution of the differential equation is

$$y = c_1 e^{t/2} \cos \frac{\sqrt{3}}{2} t + c_2 e^{t/2} \sin \frac{\sqrt{3}}{2} t, \tag{5}$$

where c_1 and c_2 are constants.

From the initial condition $y(0) = 1$, we conclude that $c_1 = 1$. From (5) and the fact that $c_1 = 1$, we obtain

$$y' = \frac{1}{2} e^{t/2} \cos \frac{\sqrt{3}}{2} t - \frac{\sqrt{3}}{2} e^{t/2} \sin \frac{\sqrt{3}}{2} t + c_2 \cdot \frac{1}{2} e^{t/2} \sin \frac{\sqrt{3}}{2} t + c_2 \cdot \frac{\sqrt{3}}{2} e^{t/2} \cos \frac{\sqrt{3}}{2} t.$$

Substituting $t = 0$ in the preceding equation and using the condition $y'(0) = 0$, we get

$$\frac{1}{2} + \frac{\sqrt{3}}{2} c_2 = 0,$$

which implies $c_2 = -1/\sqrt{3}$. Hence the solution of the given initial-value problem is

$$y = e^{t/2} \cos \frac{\sqrt{3}}{2} t - \frac{1}{\sqrt{3}} e^{t/2} \sin \frac{\sqrt{3}}{2} t.$$