MA 214 002 Calculus IV Spring 2016

Exam 2

Solutions

1. (25%) Given that $y_1(t) = t^{-1}$ is a solution of the homogeneous equation

$$t^2 y'' + 3ty' + y = 0, \qquad (t > 0)$$

use the method of reduction of order to find a second solution y_2 of the given equation. No credit will be given unless you use reduction of order to find y_2.

Solution: First we recast the given equation as

$$y'' + \frac{3}{t}y' + \frac{1}{t^2} = 0, \qquad (t > 0).$$

We seek a solution of the form $y = y_1 v = t^{-1}v$. Let $p(t) = 3/t$ and $u = v'$.

Method 1. One choice of u is given by

$$u = \frac{1}{y_1^2}\exp\left(-\int p(t)dt\right) = t^2\exp\left(-\int \frac{3}{t}\,dt\right) = t^2\exp(-3\ln t) = t^2 \cdot t^{-3} = \frac{1}{t}.$$

Since $v' = u = 1/t$, one choice of v is $v = \ln t$. Hence a second solution of the given equation is $y_2 = t^{-1}\ln t$.

Method 2. The function v satisfies the equation $tv'' + (2y_1' + py_1)v' = 0$. For $y_1 = t^{-1}$, we see that v satisfies the equation

$$t^{-1}v'' + \left(2 \cdot \frac{-1}{t^2} + \frac{3}{t}\cdot\frac{1}{t}\right)v' = 0, \qquad \text{or} \qquad t^{-1}v'' + t^{-2}v' = 0.$$

Since $u = v'$, u satisfies the equation

$$u' + \frac{1}{t}u = 0. \tag{1}$$

An integrating factor of equation (1) is:

$$\mu = e^{\int (1/t)dt} = e^{\ln t} = t.$$

Multiplying both sides of (1) by $\mu = t$, we get

$$(ut)' = 0, \qquad \text{or} \qquad u = \frac{1}{t},$$

where C is an arbitrary constant. As we need only one solution, we take $C = 1$. As $v' = u = 1/t$, we have $v = \ln t$. Therefore a second solution of the given equation is:

$$y_2 = y_1 v = t^{-1}\cdot \ln t = \frac{\ln t}{t}.$$

2. (25%) Given that $y_1 = t^{-2}$ and $y_2 = t$ constitute a fundamental set of solutions of the homogeneous equation $t^2y'' + 2ty' - 2y = 0$, where $t > 0$, use the method of variation of parameters to find a particular solution of the nonhomogeneous equation

$$t^2y'' + 2ty' - 2y = t^2. \qquad (t > 0)$$

No credit will be given unless you use the method of variation of parameters.

Solution: Note that in standard form the given equation reads:

$$y'' + \frac{2}{t}y' - \frac{2}{t^2}y = 1,$$

with the function on the right-hand side $g(t) = 1$. We seek functions u_1, u_2 such that $Y = u_1y_1 + u_2y_2$ is a particular solution of the given nonhomogeneous equation. The Wronskian of y_1, y_2 is given by

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{-2} & t \\ -2t^{-3} & 1 \end{vmatrix} = 3t^{-2}.$$

Then we have

$$u_1' = \frac{1}{W} \cdot \begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix} = \frac{t^2}{3} \cdot \begin{vmatrix} 0 & t \\ 1 & 1 \end{vmatrix} = -\frac{t^3}{3},$$
$$u_2' = \frac{1}{W} \cdot \begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix} = \frac{t^2}{3} \cdot \begin{vmatrix} t^{-2} & 0 \\ -2t^{-3} & 1 \end{vmatrix} = \frac{1}{3},$$

from which we obtain $u_1 = -t^4/12$, $u_2 = t/3$. Hence a particular solution of the given equation is:

$$Y = u_1y_1 + u_2y_2 = -\frac{t^4}{12} \cdot t^{-2} + \frac{t}{3} \cdot t = \frac{1}{4}t^2.$$

3. Consider the non-homogeneous equation

$$y'' - y' = te^t.$$

   (a) (5%) Write down the characteristic equation for the homogeneous equation $y'' - y' = 0$ and find its roots.

   (b) (10%) If the method of undetermined coefficients is to be used to find a particular solution $Y(t)$ of the given equation, give the *form* of $Y(t)$ that should be used.

   (c) (10%) Use the method of undetermined coefficients to find a particular solution of the given equation. No credit will be given unless you use the method of undetermined coefficients.

Solution: (a) The characteristic equation of the homogeneous equation $y'' - y' = 0$ is $r^2 - r = 0$, the roots of which are $r_1 = 0$, $r_2 = 1$.

(b) The right-hand side of the given non-homogeneous equation is $g(t) = te^t = P(t)e^{st}$, where the polynomial $P(t) = t$ and $s = 1$. Since $P(t)$ is of degree 1 and $s = 1$ appears once as a root of the characteristic equation, in the formula $Y(t) = t^j Q(t)e^{st}$ we have $j = 1$ and $Q(t) = At + B$, where A and B are undetermined coefficients. Therefore the form of $Y(t)$ that should be used in the method of undetermined coefficients is:

$$Y(t) = t(At + B)e^t,$$

where A and B to be determined.

(c) For $Y(t) = (At^2 + Bt)e^t$, we have $Y'(t) = (2At + B)e^t + (At^2 + Bt)e^t$, $Y''(t) = 2Ae^t + 2(2At + B)e^t + (At^2 + Bt)e^t$. Substituting $y = Y(t)$ into the given non-homogeneous equation, we obtain

$$Y'' - Y' = 2Ae^t + (2At + B)e^t = (2At + (2A + B))e^t = te^t.$$

Comparing coefficients, we conclude that

$$2A = 1, \qquad 2A + B = 0,$$

from which we get $A = 1/2$, $B = -1$. Therefore a particular solution of the given non-homogeneous equation is

$$Y(t) = t\left(\frac{1}{2}t - 1\right)e^t.$$

4. (25%) Use the Laplace transform to solve the following initial-value problem:

$$y'' + 6y' + 13y = 0, \qquad y(0) = 2, \quad y'(0) = 1.$$

No credit will be given unless you use the Laplace transform to solve the problem.

Solution: Let $Y(s) = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of both sides of the given equation, we obtain

$$s^2Y(s) - sy(0) - y'(0) + 6(sY(s) - y(0)) + 13Y(s) = 0 \qquad \text{or} \qquad (s^2 + 6s + 13)Y(s) - 2s - 13 = 0,$$

which implies

$$Y(s) = \frac{2s + 13}{s^2 + 6s + 13}.$$

We put the denominator of $Y(s)$ as a sum of squares: $s^2 + 6s + 13 = (s + 3)^2 + 12 - 3^2 = (s + 3)^2 + 2^2$. Thus we have

$$Y(s) = \frac{2(s + 3) + 13 - 6}{(s + 3)^2 + 2^2} = 2 \cdot \frac{s + 3}{(s + 3)^2 + 2^2} + \frac{7}{2}\frac{2}{(s + 3)^2 + 2^2}.$$

It follows that the solution of the given initial-value problem is:

$$y(t) = 2e^{-3t}\cos 2t + \frac{7}{2}e^{-3t}\sin 2t.$$