MA 214 Calculus IV (Spring 2016)

Section 2

Homework Assignment 10

Solutions

In what follows the Heaviside function, written as $u_c(t)$ in the text of Boyce and Diprima, is denoted by $H(t-c)$.

In each of Problems 1 through 3, find the solution of the given initial-value problem.

1. $y'' + y = H(t-\pi/2) + 3\delta(t-3\pi/2) - H(t-2\pi), \qquad y(0) = 0, \quad y'(0) = 0.$

   Solution: Taking the Laplace transform of both sides of the given equation, and using the initial conditions, we obtain

$$(s^2+1)Y(s) = \frac{e^{-\pi s/2}}{s} + 3e^{-3\pi s/2} + \frac{e^{-2\pi s}}{s}.$$

   Hence we have

$$Y(s) = \frac{e^{-\pi s/2}}{s(s^2+1)} + 3 \cdot \frac{e^{-3\pi s/2}}{s^2+1} + \frac{e^{-2\pi s}}{s(s^2+1)}.$$

   By partial fractions, we see that

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

   Therefore

$$Y(s) = e^{-\pi s/2}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 3 \cdot \frac{e^{-3\pi s/2}}{s^2+1}.$$

   It follows that the solution of the given initial-value problem is:

$$\begin{aligned} y(t) &= (1-\cos(t-\frac{\pi}{2}))H(t-\frac{\pi}{2}) + (1-\cos(t-2\pi))H(t-2\pi) + 3\sin(t-\frac{3\pi}{2})H(t-\frac{3\pi}{2}) \\ &= (1-\sin t)H(t-\frac{\pi}{2}) + (1-\cos t)H(t-2\pi) + 3\cos t\, H(t-\frac{3\pi}{2}). \end{aligned}$$

2. $2y'' + y' + 4y = \delta(t-\pi/6)\sin t, \qquad y(0) = 0, \quad y'(0) = 0.$

Solution: Taking the Laplace transform of both sides of the given equation, and using the initial conditions, we obtain

$$(2s^2+s+4)Y(s) = \int_0^\infty e^{-st}\sin t\,\delta(t-\pi/6)dt = e^{-\pi s/6}\sin\frac{\pi}{6} = \frac{1}{2}e^{-\pi s/6}.$$

Hence we have

$$Y(s) = \frac{1}{2}\cdot\frac{e^{-\pi s/6}}{2(s^2+\frac{s}{2}+2)} = \frac{1}{4}\cdot\frac{e^{-\pi s/6}}{(s+\frac{1}{4})^2+\frac{31}{16}} = \frac{1}{\sqrt{31}}\cdot e^{-\pi s/6}\cdot\frac{\sqrt{31}/4}{(s+\frac{1}{4})^2+(\frac{\sqrt{31}}{4})^2}.$$

Therefore the solution of the given initial-value problem is:

$$y(t) = \frac{1}{\sqrt{31}}H(t-\frac{\pi}{6})e^{-\frac{1}{4}(t-\frac{\pi}{6})}\sin\left(\frac{\sqrt{31}}{4}(t-\frac{\pi}{6})\right).$$

3. $y^{(4)} - y = \delta(t-1), \qquad y(0)=0, \quad y'(0)=0, \quad y''(0)=0, \quad y^{(3)}(0)=0.$

Solution: Taking the Laplace transform of both sides of the given equation, and using the initial conditions, we obtain

$$(s^4-1)Y(s) = e^{-s}.$$

Hence we have

$$\begin{aligned} Y(s) &= \frac{e^{-s}}{s^4-1} \\ &= e^{-s}\cdot\frac{1}{(s^2-1)(s^2+1)} \\ &= e^{-s}\cdot\frac{1}{2}\left(\frac{1}{s^2-1}-\frac{1}{s^2+1}\right). \end{aligned}$$

Hence the solution of the given initial-value problem is:

$$y(t) = \frac{1}{2}H(t-1)\left(\sinh(t-1)-\sin(t-1)\right).$$

4. Boyce and DiPrima, Section 6.6, p. 355, Problem 5 and Problem 10.

Solution: Problem 5. Since $f(t) = e^{-t} * \sin t$, we have

$$\mathcal{L}[f(t)] = \frac{1}{s+1}\cdot\frac{1}{s^2+1} = \frac{1}{(s+1)(s^2+1)}.$$

Problem 10. Using the formula $\mathcal{L}^{-1}[F(s)G(s)] = f(t)*g(t)$, we obtain

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2(s^2+4)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\cdot\frac{1}{s^2+4}\right] \\ &= te^{-t} * \frac{1}{2}\sin 2t = \frac{1}{2}\int_0^t (t-\tau)e^{-(t-\tau)}\sin 2\tau\,d\tau. \end{aligned}$$

5. Boyce and DiPrima, Section 6.6, p. 355, Problem 17.

Solution: Taking the Laplace transform of both sides of the given equation, and using the initial conditions, we obtain

$$(s^2+4s+4)Y(s) = 2s+5+G(s).$$

Hence we have

$$\begin{aligned} Y(s) &= \frac{2s+5}{(s+2)^2} + \frac{G(s)}{(s+2)^2} \\ &= \frac{2}{s+2} + \frac{1}{(s+2)^2} + G(s)\cdot\frac{1}{(s+2)^2}, \end{aligned}$$

where we have used the partial-fraction decomposition

$$\frac{2s+5}{(s+2)^2} = \frac{2}{s+2} + \frac{1}{(s+2)^2}.$$

The solution of the given initial-value problem is

$$\begin{aligned} y(t) &= 2e^{-2t} + te^{-2t} + te^{-2t} * g(t) \\ &= 2e^{-2t} + te^{-2t} + \int_0^t (t-\tau)e^{-2(t-\tau)}g(\tau)d\tau. \end{aligned}$$

6. Boyce and DiPrima, Section 6.6, p. 355, Problem 19.

Solution: Taking the Laplace transform of both sides of the given equation, and using the initial conditions, we obtain

$$(s^4-1)Y(s) = G(s).$$

Hence we have

$$\begin{aligned} Y(s) &= \frac{G(s)}{s^4-1} \\ &= G(s)\cdot\frac{1}{(s^2-1)(s^2+1)} \\ &= G(s)\cdot\frac{1}{2}\left(\frac{1}{s^2-1} - \frac{1}{s^2+1}\right), \end{aligned}$$

and the solution of the given initial-value problem is

$$\begin{aligned} y(t) &= \frac{1}{2}(\sinh t - \sin t) * g(t) \\ &= \frac{1}{2}\int_0^t (\sinh(t-\tau) - \sin(t-\tau))\, g(\tau)d\tau. \end{aligned}$$

7. Boyce and DiPrima, Section 6.6, p. 356, Problem 25(a).

Solution: The given integral equation is

$$\phi(t) + 2\cos t * \phi(t) = e^{-t}.$$

Let $\Phi(s) = \mathcal{L}[\phi(t)]$. Taking the Laplace transform of both sides of the given equation, we obtain

$$\Phi + 2 \cdot \frac{s}{s^2+1} \cdot \Phi = \frac{1}{s+1},$$

which implies

$$\Phi(s) = \frac{s^2+1}{(s+1)^3}.$$

By partial fractions, we find

$$\Phi(s) = \frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}.$$

Hence the solution of the given integral equation is:

$$y(t) = e^{-t} - 2te^{-t} + t^2e^{-t} = (1-t)^2e^{-t}.$$

8. Boyce and DiPrima, Section 6.6, p. 356, Problem 27(a).

Solution: The given equation is

$$\phi' - \frac{1}{2}t^2 * \phi = -t.$$

Let $\Phi(s) = \mathcal{L}[\phi(t)]$. Taking the Laplace transform of both sides of the given equation, and using the initial condition $\phi(0) = 1$, we obtain

$$s\Phi(s) - 1 - \frac{1}{2} \cdot \frac{2}{s^3} \cdot \Phi(s) = -\frac{1}{s^2},$$

or

$$\Phi(s)\left(s - \frac{1}{s^3}\right) = 1 - \frac{1}{s^2},$$

which, after some algebraic manipulations, gives

$$\Phi(s) = \frac{s}{s^2+1}.$$

Hence the solution of the given initial-value problem is $\phi(t) = \cos t$.