MA 214 Calculus IV (Spring 2016)
Section 2

Homework Assignment 1
Solutions

1. Boyce and DiPrima, p. 40, Problem 10 (c).

   Solution: In standard form the given first-order linear ODE is:

$$y' - \frac{1}{t}y = te^{-t}, \qquad t > 0.$$

An integrating factor is given by

$$\mu = e^{-\int \frac{1}{t}dt} = e^{-\ln t} = \frac{1}{t}.$$

Multiplying both sides of the ODE by μ, we obtain

$$\frac{d}{dt}\left(\frac{1}{t}y\right) = e^{-t},$$

which has the general solution

$$y(t) = -te^{-t} + Ct,$$

where C is a constant. As $t \to \infty$, we see that $y(t) \to \infty$ if $C > 0$, $y(t) \to 0$ if $C = 0$, and $y(t) \to -\infty$ if $C < 0$.

2. Boyce and DiPrima, p. 40, Problem 18.

   Solution: First we put the given linear first-order ODE in standard form:

$$y' + \frac{2}{t}y = \frac{\sin t}{t}, \qquad t > 0.$$

An integrating factor is given by

$$\mu = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2.$$

Multiplying both sides of the ODE by μ, we obtain

$$\frac{d}{dt}\left(t^2 y\right) = t\sin t.$$

Integrating both sides of the preceding equation, we get

$$\begin{aligned} t^2 y(t) &= \int t \sin t \, dt \\ &= \int t \, d(-\cos t) = -t\cos t + \int \cos t \, dt \\ &= -t\cos t + \sin t + C, \end{aligned}$$

where C is a constant. From the initial condition $y(\pi/2) = 1$, we deduce that $C = (\pi/2)^2 - 1$. Hence the solution to the given initial-value problem is:

$$y = \frac{1}{t^2}\left(\frac{\pi^2}{4} - 1 + \sin t - t\cos t\right).$$

3. Boyce and DiPrima, p. 40, Problem 20.

   Solution: In standard form the given equation reads:

$$y' + \left(1 + \frac{1}{t}\right) y = 1, \qquad t > 0.$$

An integrating factor is given by

$$\mu = \exp\left(\int \left(1 + \frac{1}{t}\right) dt\right) = e^{t + \ln t} = e^t \cdot e^{\ln t} = te^t.$$

Multiplying both sides of the ODE by μ, we obtain

$$\frac{d}{dt}\left(t\, e^t y\right) = t\, e^t.$$

Integrating both sides of the preceding equation with respect to t yields

$$t\, e^t y = t\, e^t - e^t + C,$$

or

$$y = 1 - \frac{1}{t} + \frac{Ce^{-t}}{t},$$

where C is a constant to be determined by the initial condition. From the initial condition that $y(\ln 2) = 1$, we obtain $C = 2$. Hence the solution to the given initial-value problem is:

$$y = 1 - \frac{1}{t} + \frac{2e^{-t}}{t}.$$

4. Boyce and DiPrima, p. 40, Problem 28.

   Solution: It is easy to solve the initial-value problem

$$y' + \frac{2}{3}y = 1 - \frac{1}{2}t, \qquad y(0) = y_o$$

   to get the solution

$$y(t) = \frac{21}{8} - \frac{3}{4}t + (y_o - \frac{21}{8})e^{-2t/3}.$$

   Indeed, $\mu = e^{2t/3}$ is an integrating factor. Multiplying both sides of the given equation by μ, we obtain

$$(ye^{2t/3})' = e^{2t/3} - \frac{1}{2}te^{2t/3}.$$

   Integrating both sides of the preceding equation, we get

$$\begin{aligned} ye^{2t/3} &= \frac{3}{2}e^{2t/3} - \frac{1}{2}\int te^{2t/3}\,dt \\ &= \frac{3}{2}e^{2t/3} - \frac{1}{2}\left(t \cdot \frac{3}{2}e^{2t/3} - \frac{9}{4}e^{2t/3}\right) + C, \end{aligned}$$

   where we have used integration by parts. Using the initial condition to put C in terms of y_o leads to the solution given above.

   If there is a $t = \tau$ at which the solution touches the t-axis but does not cross it, then at $t = \tau$ we have $y(\tau) = 0$ and $y'(\tau) = 0$. From the given differential equation, we observe that $\tau = 2$. It follows that

$$y(2) = \frac{21}{8} - \frac{3}{2} + (y_o - \frac{21}{8})e^{-4/3} = 0,$$

   from which we obtain

$$y_o = \frac{21 - 9e^{4/3}}{8} = -1.642876.$$

5. Boyce and DiPrima, p. 41, Problem 30.

   Solution: Multiplying both sides of the ODE

$$y' - y = 1 + 3\sin t$$

   by the integrating factor $\mu = e^{-t}$, we obtain

$$(e^{-t}y)' = e^{-t} + 3e^{-t}\sin t.$$

   Integrating both sides of the preceding equation, we get

$$e^{-t}y = -e^{-t} + 3\int e^{-t}\sin t\,dt = -e^{-t} + 3I, \qquad (1)$$

where $I = \int e^{-t} \sin t \, dt$. Using integration by parts twice, we have

$$I = \int e^{-t} d(-\cos t) = -e^{-t} \cos t - \int e^{-t} \cos t \, dt$$
$$= -e^{-t} \cos t - \left(e^{-t} \sin t + \int e^{-t} \sin t \, dt \right) = -e^{-t}(\sin t + \cos t) - I + C_1.$$

Hence $I = -\frac{1}{2}\left(e^{-t}(\sin t + \cos t) + C_1\right)$. Substituting this expression of I into (1), we obtain the general solution of the given differential equation:

$$y = -1 - \frac{3}{2}\left(e^{-t}(\sin t + \cos t)\right) + Ce^t,$$

where C is a constant. Imposing the initial condition $y(0) = y_o$ leads to the solution of the given initial-value problem:

$$y(t) = -1 - \frac{3}{2}(\sin t + \cos t) + \left(y_o + \frac{5}{2}\right) e^t.$$

For a bounded solution, we must have $y_o = -5/2$.

6. Boyce and DiPrima, p. 48, Problem 2.

Solution: From $y' = \dfrac{x^2}{y(1+x^3)}$, we get

$$ydy = \frac{x^2}{1+x^3} dx.$$

It follows that the general solution is:

$$\frac{y^2}{2} = \frac{1}{3} \ln |1 + x^3| + C.$$

The given differential equation requires that $y \neq 0$ and $x \neq -1$.

7. Boyce and DiPrima, p. 48, Problem 16 (a), (c).

Solution: From $y' = \dfrac{x(x^2+1)}{4y^3}$, we get

$$4y^3 dy = (x^3 + x)dx,$$

which has the general solution

$$y^4 = \frac{x^4}{4} + \frac{x^2}{2} + C.$$

The initial condition $y(0) = -1/\sqrt{2}$ dictates that $C = 1/4$. Hence we have

$$y^4 = \frac{x^4 + 2x^2 + 1}{4} = \frac{(x^2+1)^2}{4}.$$

It follows that the solution to the given initial-value problem is:

$$y = -\sqrt{\frac{x^2+1}{2}}.$$

The domain of the solution is clearly $(-\infty, \infty)$.

8. Boyce and DiPrima, p. 49, Problem 22.

   Solution: The given equation $y' = 3x^2/(3y^2 - 4)$ is separable. Separating the variables and integrating both sides of the equation, we get

   $$\int (3y^2 - 4)\, dy = \int 3x^2\, dx + C$$

   or

   $$y^3 - 4y = x^3 + C,$$

   where C is a constant to be determined from the initial condition. From the initial condition $y(1) = 0$, we obtain $C = -1$. Hence the required solution is given implicitly by the equation

   $$y^3 - 4y = x^3 - 1.$$

   A glance at the given differential equation reveals that $y' \to \pm\infty$ as $y \to \pm 2/\sqrt{3}$. When $y = 2/\sqrt{3}$, $x = [1 - 16/(3\sqrt{3})]^{1/3} \approx -1.276$; when $y = -2/\sqrt{3}$, $x = [1 + 16/(3\sqrt{3})]^{1/3} \approx 1.598$. Hence the approximate interval on which the solution is defined is $(-1.276, 1.598)$, which contains the point $x = 1$.

9. Boyce and DiPrima, p. 49, Problem 24.

   Solution: It is easy to solve the initial-value problem

   $$y' = \frac{2 - e^x}{3 + 2y}, \qquad y(0) = 0.$$

   The solution is

   $$3y + y^2 = 2x - e^x + 1$$

   or in explicit form

   $$y = -\frac{3}{2} + \sqrt{\frac{13}{4} + 2x - e^x}.$$

   The solution y assumes its maximum value when the function $f(x) = 13/4 + 2x - e^x$ is at its absolute maximum. Note that $f'(x) = 0$ implies $2 = e^x$ or $x = \ln 2$, and $f''(\ln 2) = -2 < 0$. Hence f has only one local maximum, which is located at $x = \ln 2$. Since $f(\ln 2) = 13/4 + 2(\ln 2 - 1) > f(0) = 9/4$, f attains its maximum value at $x = \ln 2$.

10. Boyce and DiPrima, p. 51, Problem 38.

Solution: (a) The given differential equation can be put in the form

$$\frac{dy}{dx} = f(y/x), \qquad \text{where } f(y/x) = \frac{3(y/x)^2 - 1}{2(y/x)}.$$

Hence the given differential equation is homogeneous.

(b) The substitution $y = vx$ reduces the given differential equation to the form

$$v'x + v = \frac{3}{2}v - \frac{1}{2v},$$

which is equivalent to

$$x\frac{dv}{dx} = \frac{1}{2}\left(v - \frac{1}{v}\right) = \frac{1}{2} \cdot \frac{v^2 - 1}{v}.$$

It is easy to see that $v = 1$ and $v = -1$ are special solutions of the preceding differential equation. To seek other solutions (i.e., $v \neq \pm 1$), we put the separable equation in the form

$$\frac{2v}{v^2 - 1}dv = \frac{dx}{x}.$$

The general solution to the preceding equation is:

$$\ln|v^2 - 1| = \ln|x| + C_1 \quad \text{or} \quad \ln\left|\frac{y^2 - x^2}{x^3}\right| = C_1,$$

which can be put in the form

$$\left|\frac{y^2 - x^2}{x^3}\right| = C, \qquad \text{where } C = e^{C_1} \text{ is a positive constant.}$$

Note that if we put $C = 0$, then we get $y = \pm x$, which are none other than the special solutions $v = \pm 1$. Thus the formula $|y^2 - x^2| = C|x^3|$, where the constant $C \geq 0$, includes all solutions of the given differential equation.