MA 214 Calculus IV (Spring 2016)

Section 2

Homework Assignment 2

Solutions

1. Boyce and DiPrima, p. 60, Problem 2.

   Solution: Let $M(t)$ be the mass (in grams) of salt in the tank after t minutes. The initial-value problem that governs $M(t)$ is:

   $$\frac{dM}{dt} = 2\gamma - \frac{M}{120} \cdot 2, \qquad M(0) = 0.$$

   The ODE is linear. Its solution is

   $$M(t) = 120\gamma + Ce^{-t/60}.$$

   From the initial condition $M(0) = 0$, we deduce that $C = -120\gamma$. Hence we have

   $$M(t) = 120\gamma \left(1 - e^{-t/60}\right) \text{ grams.}$$

   As $t \to \infty$, we see that $M(t) \to 120\gamma$ grams.

2. A tank initially contain 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus the tank is empty after exactly 1 hour.

   (a) Find the amount of salt in the tank after t minutes.

   (b) What is the maximum amount of salt ever in the tank?

   Solution: Let $V(t)$ be the volume of the solution (in gal) and $M(t)$ the mass of salt (in lb) in the tank after t minutes. Clearly $V(t) = 60 - t$ gal. The initial-value problem that governs $M(t)$ is:

   $$\frac{dM}{dt} = 2 - \frac{M}{60 - t} \cdot 3, \qquad M(0) = 0.$$

   The first-order ODE in question is linear, and

   $$\mu = e^{\int \frac{3}{60-t} dt} = e^{-3\ln(60-t)} = (60 - t)^{-3}$$

is an integrating factor. Multiplying both sides of the ODE by μ, we obtain

$$\frac{d}{dt}\big((60-t)^{-3}M(t)\big) = 2(60-t)^{-3},$$

which implies

$$(60-t)^{-3}M(t) = (60-t)^{-2} + C,$$

where C is a constant. From the initial condition $M(0) = 0$, we find $C = -1/60^2$. It follows that

$$M(t) = (60-t) - \frac{1}{60^2}(60-t)^3,$$

which is the mass of salt (in lb) in the tank after t minutes.

To find the maximum amount of salt ever in the tank, we find M' and set it equal to zero:

$$M' = -1 + \frac{3}{60^2}(60-t)^2 = 0,$$

which implies $t = 60(1 - 1/\sqrt{3})$. When $M' = 0$, we see from the ODE and $t = 60(1 - 1/\sqrt{3})$ that

$$M = \frac{2}{3}(60-t) = \frac{2}{3}\cdot\frac{60}{\sqrt{3}} = \frac{40}{\sqrt{3}} \text{ lb}.$$

Since $M(0) = 0$, $M(60) = 0$, and $M(\cdot)$ is a continuous function on the interval $[0, 60]$, $M(\cdot)$ attains its absolute maximum value at some point in $[0, 60]$. At this instant, $M' = 0$. Since there is only one t at which $M' = 0$, we must have $t = 60(1 - 1/\sqrt{3})$, and the maximum amount of salt ever in the tank is $40/\sqrt{3}$ lb.

3. A 100-gallon mixing vat is initially full of pure water. One gallon per minute of salt solution with 1 pound of salt dissolved in each gallon of water flows into the tank, 1 gallon per minute of mixed solution flows out, and 1 gallon of water per minute evaporates from the vat. Estimate the amount of salt in the vat when it is half empty.

   Solution: Let $V(t)$ be the volume of the solution (in gal) and $M(t)$ the mass of salt (in lb) in the tank after t minutes. Clearly $V(t) = 100 - t$ gal. The initial-value problem that governs $M(t)$ is:

$$\frac{dM}{dt} + \frac{M}{100-t} = 1, \qquad M(0) = 0.$$

   The first-order ODE in question is linear, and

$$\mu = e^{\int \frac{1}{100-t}dt} = e^{-\ln(100-t)} = (100-t)^{-1}$$

   is an integrating factor. Multiplying both sides of the ODE by μ, we obtain

$$\frac{d}{dt}\big((100-t)^{-1}M(t)\big) = (100-t)^{-1},$$

which implies

$$(100-t)^{-1}M(t) = -\ln(100-t) + C,$$

where C is a constant. From the initial condition $M(0) = 0$, we get $C = \ln 100$. Hence the solution of the initial-value problem is:

$$M(t) = (100-t)\ln\left(\frac{100}{100-t}\right).$$

The instant at which the vat is half-empty is $t = 50\,\text{min}$. The mass of salt in the vat at that instant is:

$$M(50) = 50(\ln 2)\text{ lb}.$$

4. Boyce and DiPrima, p. 61, Problem 10.

Solution: Let $S(t)$ be the amount the home buyer owes after t years. The initial-value problem that $S(t)$ satisfies is:

$$\frac{dS}{dt} = rS - k, \qquad S(0) = S_0,$$

where S_0 is the amount borrowed, $r = 0.06$ is the annual interest rate, and $k = \$18{,}000$ per year is the annual rate of payment. The solution of the initial-value problem in question is

$$S(t) = S_0 e^{rt} - \frac{k}{r}(e^{rt} - 1).$$

(a) The maximum S_0 that can be paid off after 20 years is given by the equation

$$S_0 e^{1.2} - \frac{18000}{0.06}(e^{1.2} - 1) = 0,$$

which gives $S_0 = \$209{,}641.74$.

The maximum S_0 that can be paid off after 30 years is given by the equation

$$S_0 e^{1.8} - \frac{18000}{0.06}(e^{1.8} - 1) = 0,$$

which gives $S_0 = \$250{,}410.33$.

(b) The total interest paid on the 20-year mortgage is $20 \times \$18,000 - \$209,641.74$ or \$150,358.26.

The total interest paid on the 30-year mortgage is $30 \times \$18,000 - \$250,410.33$ or \$289,589.67.

5. Boyce and DiPrima, p. 62, Problem 12.

Solution: Let $S(t)$ be the amount the college graduate owes after t months. The initial-value problem that $S(t)$ satisfies is:

$$\frac{dS}{dt} = rS - (k + \alpha t), \qquad S(0) = S_0,$$

where S_0 is the amount borrowed, $r = 0.005$ is the monthly interest rate, and $k + \alpha t = 800 + 10t$ is the monthly rate of payment. The solution of the initial-value problem in question is

$$S(t) = S_0 e^{rt} + \left(\frac{k}{r} + \frac{\alpha}{r^2}\right)(1 - e^{rt}) + \frac{\alpha t}{r}.$$

(a) The number of months of payment required to cover the loan is found by solving numerically the equation

$$5.6 \times 10^5 - 4.1 \times 10^5 \exp(0.005\, t) + 2 \times 10^3 t = 0.$$

The answer is 146.54 months.

(b) The loan amount that could be paid off in exactly 20 years is given by the equation

$$S_0 \times \exp(0.005 \times 240) + 5.6 \times 10^5 (1 - \exp(0.005 \times 240) + 2 \times 10^3 \times 240 = 0.$$

The answer is \$246,758.

6. Boyce and DiPrima, p. 63, Problem 16.

Solution: Let $T(t)$ be the temperature (in F) of the coffee after t minutes, T_o the initial temperature of the coffee, and T_e the temperature of the room. The initial-value problem that governs the cooling of the coffee is given by

$$\frac{dT}{dt} = -k(T - T_e), \qquad T(0) = T_o,$$

where $k > 0$ is a constant. As shown in class, the solution of this initial-value problem is:

$$T(t) = (T_o - T_e)e^{-kt} + T_e.$$

From the given data that $T_o = 200\,\mathrm{F}$, $T_e = 70\,\mathrm{F}$, and that the coffee is at $190\,\mathrm{F}$ after 1 minute, we have

$$190 = 130e^{-k} + 70,$$

from which we get $k = \ln(13/12)\ \mathrm{min}^{-1}$. When $T(t) = 150\,\mathrm{F}$, we have

$$150 = 130e^{-t\ln(13/12)},$$

which implies

$$t = \frac{\ln(13/8)}{\ln(13/12)} \approx 6.07\ \mathrm{min}.$$

7. Boyce and DiPrima, p. 64, Problem 19 (a), (b).

   Solution: By conservation of mass, the differential equation that governs the amount $Q(t)$ of pollutants at time t is:

$$\frac{dQ}{dt} = kr - \frac{Q}{V}r + P.$$

   Substituting $Q(t) = c(t)V$ into the preceding equation and putting the resulting equation in standard form, we obtain

$$\frac{dc}{dt} + \frac{r}{V}c = \frac{kr + P}{V}. \tag{1}$$

   (a) Solving (1) with initial condition $c(0) = c_0$, we get

$$c(t) = \left(c_0 - (k + \frac{P}{r})\right) e^{-rt/V} + \left(k + \frac{P}{r}\right).$$

   It is easy to see that $\lim_{t\to\infty} c(t) = k + P/r$.

   (b) If $k = 0$ and $P = 0$ for $t > 0$, then the concentration of pollutants is given by

$$c(t) = c_0 e^{-rt/V}.$$

   For the concentration of pollutants to reduce to $c_0/2$, the elapsed time T is obtained from the equation

$$\frac{c_0}{2} = c_0 e^{-rT/V},$$

   which gives $T = (V \ln 2)/r$.

   For the concentration of pollutants to reduce to $c_0/10$, the elapsed time T is obtained from the equation

$$\frac{c_0}{10} = c_0 e^{-rT/V},$$

   which gives $T = (V \ln 10)/r$.

8. Boyce and DiPrima, p. 65, Problem 25 (a).

   Solution: We choose a Cartesian coordinate system such that X-axis is normal to the ground and is pointing upward and the origin is at ground level. By Newton's second law, the differential equation that governs the motion of the projected body is:

$$m\frac{dv}{dt} = -mg - kv, \quad \text{or} \quad \frac{dv}{dt} = -g - \frac{kv}{m}.$$

   To determine the maximum height x_m attained by the body, we use the relation $dv/dt = v(dv/dx)$ and put the equation of motion in the form

$$v\frac{dv}{dx} = -g - \frac{kv}{m}.$$

The preceding equation is separable, and we recast it as

$$\left(1 - \frac{mg/k}{v + mg/k}\right) dv = -\frac{k}{m} dx. \tag{2}$$

When $t = 0$, we have $x = 0$ and $v = v_0$; when $t = t_m$, we have $x = x_m$ and $v = 0$. We integrate the left-hand side of (2) from $v = v_0$ to $v = 0$ and the right-hand side from $x = 0$ to $x = x_m$, both of which correspond to integration over the time interval $[0, t_m]$. We obtain the equation

$$\int_{v_0}^{0} \left(1 - \frac{mg/k}{v + mg/k}\right) dv = \int_{0}^{x_m} -\frac{k}{m} dx.$$

Thus we get the following equation for x_m:

$$\left[v - \frac{mg}{k} \ln\left(v + \frac{mg}{k}\right)\right]_{v_0}^{0} = -\frac{k}{m} x_m.$$

Simplifying the preceding equation, we arrive at the following expression for the maximum height attained by the projected body:

$$x_m = \frac{mv_0}{k} - \frac{m^2 g}{k^2} \ln\left(1 + \frac{kv_0}{mg}\right).$$

To find t_m, the time at which the maximum height is reached, we solve the initial-value problem

$$\frac{dv}{dt} = -g - \frac{kv}{m}, \qquad v(0) = v_0.$$

Note that the differential equation in question is linear. The solution of the initial-value problem is:

$$v = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-kt/m}.$$

Since $v = 0$ when $t = t_m$, t_m is given by the following equation:

$$0 = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-kt_m/m},$$

from which we get

$$t_m = \frac{m}{k} \ln\left(1 + \frac{kv_0}{mg}\right).$$