MA 214 Calculus IV (Spring 2016)

Section 2

Homework Assignment 3

Solutions

1. Boyce and DiPrima, p. 75, Problems 3 and 6.

   Solution: Problem 3. Note that the interval which contains π and on which $p(t) = \tan t$ is continuous is $(\pi/2, 3\pi/2)$, and $g(t) = \sin t$ is continuous on $(-\infty, \infty)$. By Theorem 2.4.1, there exists a unique solution of the given initial-value problem on $(\pi/2, 3\pi/2)$.

   Problem 6. First we put the given ODE in standard form: For $t > 0$ and $t \neq 1$,

   $$y' + \frac{1}{\ln t} y = \frac{\cot t}{\ln t}.$$

   The interval which contains 2 and on which $p(t) = 1/\ln t$ and $g(t) = \cot t/\ln t$ are continuous is $(1, \infty) \cap (0, \pi)$. Hence by Theorem 2.4.1, the given initial-value problem has a unique solution on $(1, \pi)$.

2. Boyce and DiPrima, p. 76, Problems 9 and 10.

   Solution: Problem 9. Let $f(t, y) = \dfrac{\ln |ty|}{1 - t^2 + y^2}$. Then

   $$\frac{\partial f}{\partial y} = \frac{(1 - t^2 + y^2) - 2y^2 \ln |ty|}{y(1 - t^2 + y^2)^2}.$$

   Both f and $\partial f/\partial y$ are continuous on the open region

   $$\Omega = \{(t, y) \in \mathbb{R}^2 : t \neq 0, y \neq 0, 1 - t^2 + y^2 \neq 0\},$$

   which is the region where the hypotheses of Theorem 2.4.2 are satisfied.

   Problem 10. Let $f(t, y) = (t^2 + y^2)^{3/2}$. Then $\partial f/\partial y = 3y(t^2 + y^2)^{1/2}$. Both f and $\partial f/\partial y$ are continuous on the entire ty-plane. Hence the hypotheses of Theorem 2.4.2 are satisfied everywhere in the ty-plane.

3. Boyce and DiPrima, p. 76, Problem 14.

   Solution: For $y_o = 0$, clearly $y = 0$, which is defined on $(-\infty, \infty)$, is the unique solution of the initial-value problem in question.

For $y_o \neq 0$, by the uniqueness theorem the solution curve will never meet the line $y = 0$, i.e., $y(t) \neq 0$ for all t. By dividing both sides of the given ODE by y^2, we separate the variables and obtain

$$\frac{1}{y^2}dy = 2tdt, \qquad \text{or} \qquad -\frac{1}{y} = t^2 + C.$$

The initial condition $y(0) = y_o$ dictates that $C = -1/y_o$. Hence the solution of the initial-value problem is

$$y = \frac{-1}{t^2 - 1/y_o}.$$

For $y_o < 0$, we have $t^2 - 1/y_o > 0$, and the solution is defined on $(-\infty, \infty)$.

For $y_o > 0$, the solution cannot cross the lines $t = \pm\sqrt{1/y_o}$. Since $0 \in (-1/\sqrt{y_o}, 1/\sqrt{y_o})$, the interval of existence of the solution is $(-1/\sqrt{y_o}, 1/\sqrt{y_o})$.

4. Boyce and DiPrima, p. 77, Problem 22.

Solution: Let $f(t, y) = \dfrac{-t + (t^2 + 4y)^{1/2}}{2}$.

(a) For $y_1(t) = 1 - t$, we have $y_1' = -1$ and

$$f(t, 1-t) = \frac{-t + \sqrt{(t-2)^2}}{2} = \begin{cases} \frac{-t+(t-2)}{2} \\ \frac{-t+(2-t)}{2} \end{cases} = \begin{cases} -1 & \text{for } t \geq 2 \\ 1 - t & \text{for } t < 2. \end{cases}$$

Hence y_1 is a solution of the given initial-value problem for $t \geq 2$. For $y_2(t) = -t^2/4$, we have $y_2' = -t/2$ and $f(t, -t^2/4) = -t/2$ for all t; moreover, $y_2(2) = -1$. Hence y_2 is a solution of the given initial-value problem for all t.

(b) Since $\dfrac{\partial f}{\partial y} = \dfrac{1}{\sqrt{t^2 + 4y}}$, which is not continuous in any rectangular box containing the point $(t, y) = (2, -1)$, the hypotheses of Theorem 2.4.2 are not all satisfied. Hence the existence of two solutions of the given problem does not contradict Theorem 2.4.2.

(c) For $y = ct + c^2$, where c is a constant, we have $y'(t) = c$ and

$$f(t, ct + c^2) = \frac{-t + \sqrt{(t + 2c)^2}}{2} = c$$

for $t \geq -2c$, and $f(t, ct + c^2) = -(t + c)$ for $t < -2c$. Hence $y = ct + c^2$ satisfies the given differential equation for $t \geq -2c$. It is obvious that if $c = -1$, the initial condition is also satisfied, and the solution $y = y_1(t)$ is obtained.

If $y = ct + c^2 = y_2(t) = -t^2/4$, then $ct + c^2 = -t^2/4$ or $(t + 2c)^2 = 0$, which implies $t = -2c$. This is impossible because t is a variable and c is a constant. Therefore there is no choice of constant c which makes $y = ct + c^2 = y_2(t)$.

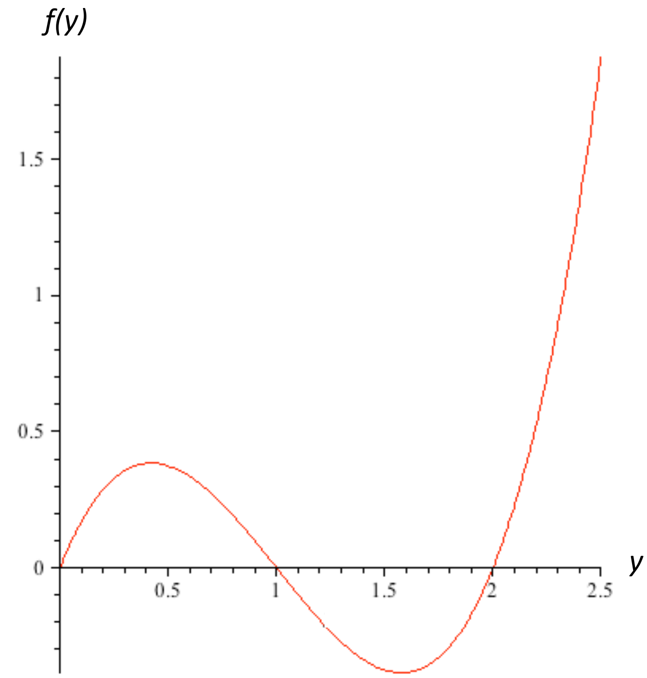

Figure 1: Sketch of the graph of $f(y) = y(y-1)(y-2)$.

5. Boyce and DiPrima, p. 88, Problem 3.

   Solution: Here $dy/dt = f(y) = y(y-1)(y-2), y_0 \geq 0$. Hence the critical points, at which $f(y) = 0$, are: $y = 0, 1, 2$. A sketch of the graph of f is given in Figure 1.

   The critical points divide $(0, \infty)$ into three open intervals, namely: $(0, 1)$, $(1, 2)$, and $(2, \infty)$. It is easy to see that $y' = f(y) > 0$ and $y(t)$ is increasing for $y \in (0, 1)$ and $y \in (2, \infty)$; $y' = f(y) < 0$ and $y(t)$ is decreasing for $y \in (1, 2)$. See Figure 1. The phase line of the system modeled by the given differential equation (with $y_0 \geq 0$) is shown in Figure 2. It follows that the critical points $y = 0$, $y = 1$, and $y = 2$ are unstable, asymptotically stable, and unstable, respectively.

   To determine the concavity of solution curves, we examine the sign of $y'' = f(y)f'(y)$.

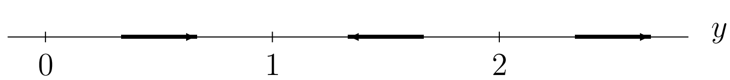

Figure 2: Phase line of system modeled by $y' = y(y-1)(y-2)$, $y_0 \geq 0$.

Intervals for y	$(0,a)$	$(a,1)$	$(1,b)$	$(b,2)$	(b,∞)
$y' = f(y)$	$+$	$+$	$-$	$-$	$+$
$f'(y)$	$+$	$-$	$-$	$+$	$+$
$y'' = f(y)f'(y)$	$+$	$-$	$+$	$-$	$+$
Concavity	CU	CD	CU	CD	CU

Table 1: Concavity of solution curves on various intervals for y.

By direct differentiation, we find $f'(y) = 3y^2 - 6y + 2$ and f assumes a local maximum at $a = 1 - \sqrt{3}/3$ and a local minimum at $b = 1 + \sqrt{3}/3$. Thus $f'(y) > 0$ on the intervals $(0,a)$ and (b,∞); $f'(y) < 0$ on the interval (a,b). From the sign of $f(y)$ and of $f'(y)$, we infer the concavity of solution curves on various intervals for y; see Table 1.

Representative solution curves will be sketched on the board in class on Wednesday, 9/23.

6. Boyce and DiPrima, p. 89, Problem 8.

   Solution: Here $dy/dt = f(y) = -k(y-1)^2$, $k > 0, -\infty < y_0 < \infty$. There is only one critical point: $y = 1$. A sketch of the graph of f for the case $k = 2$ is given in Figure 3.

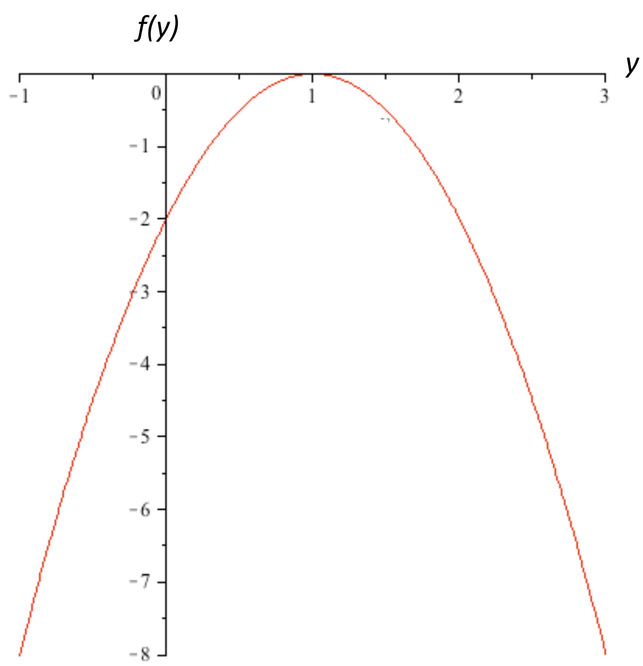

Figure 3: Sketch of the graph of $f(y) = -k(y-1)^2$, where k is taken as 2.

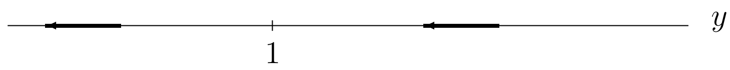

Figure 4: Phase line of system modeled by $y' = -k(y-1)^2$, $k > 0$.

The critical point $y = 1$ divides $(-\infty, \infty)$ into two open intervals: $(-\infty, 1)$ and $(1, \infty)$, on both of which $y' = f(y) < 0$. The phase line in question is shown in Figure 4. It is clear that the critical point $y = 1$ is semi-stable.

Since $f'(y) = -2k(y-1)$, clearly $f'(y) > 0$ for $y \in (-\infty, 1)$, and $f'(y) < 0$ for $y \in (1, \infty)$. It follows that $y''(t) = f(y)f'(y) < 0$ and solution curves are concave down when $y \in (-\infty, 1)$; $y''(t) = f(y)f'(y) > 0$ and solution curves are concave up when $y \in (1, \infty)$.

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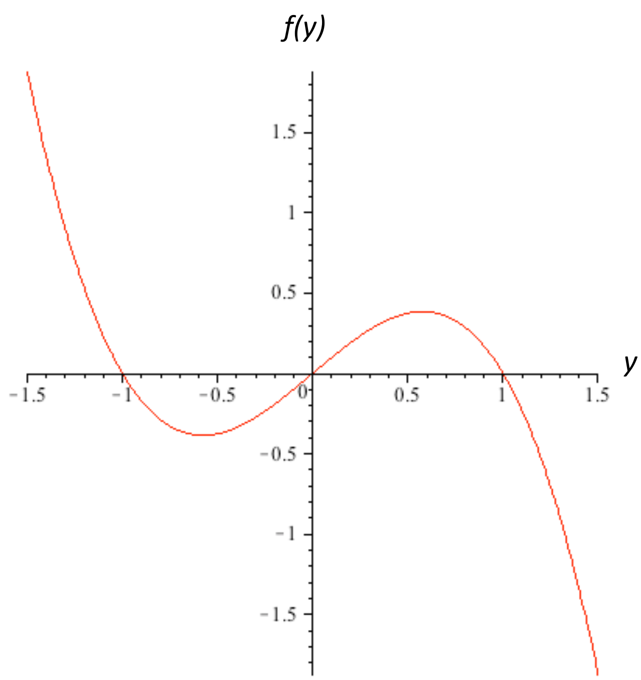

Figure 5: Sketch of the graph of $f(y) = y(1-y^2)$.

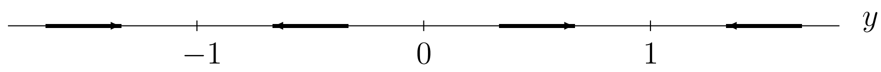

Figure 6: Phase line of system modeled by $y' = y(1-y^2)$, $-\infty < y_0 < \infty$.

Intervals for y	$(-\infty,-1)$	$(-1,-1/\sqrt{3})$	$(-1/\sqrt{3},0)$	$(0,1/\sqrt{3})$	$(1/\sqrt{3},1)$	$(1,\infty)$
$y' = f(y)$	$+$	$-$	$-$	$+$	$+$	$-$
$f'(y)$	$-$	$-$	$+$	$+$	$-$	$-$
$y'' = f(y)f'(y)$	$-$	$+$	$-$	$+$	$-$	$+$
Concavity	CD	CU	CD	CU	CD	CU

Table 2: Concavity of solution curves on various intervals for y.

7. Boyce and DiPrima, p. 89, Problem 10.

Solution: Here $dy/dt = f(y) = y(1-y^2)$, $-\infty < y_0 < \infty$. The critical points are clearly: $y = -1, 0, 1$. A sketch of the graph of f is given in Figure 5.

The critical points divide $(-\infty,\infty)$ into four open intervals, namely: $(-\infty,-1)$, $(-1,0)$, $(0,1)$, and $(1,\infty)$. It is easy to see that $y' = f(y) > 0$ and $y(t)$ is increasing for $y \in (-\infty,-1)$ and $y \in (0,1)$; $y' = f(y) < 0$ and $y(t)$ is decreasing $y \in (-1,0)$ and $y \in (1,\infty)$. See Figure 5. The phase line of the system modeled by the given differential equation is shown in Figure 6. It follows that the critical points $y = -1$, $y = 0$, and $y = 1$ are asymptotically stable, unstable, and asymptotically stable, respectively.

To determine the concavity of solution curves, we examine the sign of $y'' = f(y)f'(y)$. By direct differentiation, we find $f'(y) = 1 - 3y^2$ and f assumes a local minimum at $y = -1/\sqrt{3}$ and a local maximum at $y = 1/\sqrt{3}$. Thus $f'(y) < 0$ on the intervals $(-\infty,-1/\sqrt{3})$ and $(1/\sqrt{3},\infty)$; $f'(y) > 0$ on the interval $(-1/\sqrt{3},1/\sqrt{3})$. From the sign of $f(y)$ and of $f'(y)$, we infer the concavity of solution curves on various intervals for y; see Table 2.

Representative solution curves will be sketched on the board in class on Wednesday, 9/23.

8. Boyce and DiPrima, p. 90, Problem 18.

Solution: (a) Let $V(t)$ and $A(t)$ be the volume and surface area of water in the conical pond at time t, respectively. Let $r(t)$ be the radius of the water surface and $d(t)$ be the depth of the water at the center of the pond. Then we have

$$d(t) = \frac{hr(t)}{a}, \qquad \text{and} \qquad V(t) = \frac{1}{3}\pi r^2(t)d(t) = \left(\frac{\pi h}{3a}\right) r^3(t),$$

which imply

$$r(t) = \left(\frac{3aV(t)}{\pi h}\right)^{1/3}, \qquad \text{and} \qquad A(t) = \pi\left(\frac{3a}{\pi h}\right)^{2/3} V^{2/3}(t).$$

Hence, by the given hypotheses, $V(t)$ satisfies the differential equation

$$\frac{dV}{dt} = k - \alpha A = k - \pi\left(\frac{3a}{\pi h}\right)^{2/3} V^{2/3},$$

where α is the coefficient of evaporation.

(b) In what follows we use the suffix "eq" to denote the equilibrium value of a quantity. The equilibrium depth d_{eq} of water in the pond is determined by the condition $V' = 0$ or $A_{\text{eq}} = k/\alpha$. Since

$$A_{\text{eq}} = \pi r_{\text{eq}}^2 = \pi(ad_{\text{eq}}/h)^2,$$

the equilibrium depth is given by

$$d_{\text{eq}} = \sqrt{\frac{k}{\alpha\pi}} \cdot \frac{h}{a}.$$

Note that $V' < 0$ and $V' > 0$ when $V > V_{\text{eq}}$ and $V < V_{\text{eq}}$, respectively. Hence the equilibrium is asymptotically stable.

(c) The pond will not overflow if $d_{\text{eq}} \leq h$ or $k \leq \alpha\pi a^2$.