MA 214 Calculus IV (Spring 2016)
Section 2

Homework Assignment 6
Solutions

1. Boyce and DiPrima, p. 151, Problem 12.

   Solution: First we put the given differential equation in standard form:

$$y'' + \frac{1}{x-2}y' + \tan x = 0.$$

   The function $p(x) = 1/(x-2)$ is continuous on $(-\infty, 2) \cup (2, \infty)$, whereas $x = (2k+1)\pi/2$, where $k = 0, \pm 1, \pm 2, \cdots$, are the points of discontinuity of the function $q(x) = \tan x$. The largest interval which contains $x_o = 3$ and on which both p and q are continuous is $(2, 3\pi/2)$. By Theorem 3.2.1, the longest interval on which the given initial-value problem is certain to have a unique solution is $(2, 3\pi/2)$.

2. Boyce and DiPrima, p. 156, Problem 17.

   Solution: We have

$$W(f,g) = \begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix} = e^{2t} \begin{vmatrix} 1 & g \\ 2 & g' \end{vmatrix} = 3e^{4t}.$$

   Hence g satisfies the differential equation

$$g' - 2g = 3e^{2t}.$$

   An integrating factor of the preceding linear equation is $\mu(t) = e^{-2t}$. Hence we have $(e^{-2t}g)' = 3$, which implies

$$e^{-2t}g(t) = 3t + C, \qquad \text{or} \qquad g(t) = 3te^{2t} + Ce^{2t},$$

   where C is an arbitrary constant.

3. Boyce and DiPrima, p. 156, Problem 23.

   Solution: We seek solutions y_1 and y_2 of the differential equation

$$y'' + 4y' + 3y = 0$$

that satisfy the initial conditions

$$y_1(1) = 1, \quad y_1'(1) = 0,$$

and

$$y_2(1) = 0, \quad y_2'(1) = 1,$$

respectively. The characteristic equation of the given differential equation is $r^2 + 4r + 3 = 0$, which has two real roots $r_1 = -1$ and $r_2 = -3$. Hence the general solution is

$$y = c_1 e^{-t} + c_2 e^{-3t}.$$

From the initial conditions for solution y_1, we obtain the simultaneous equations

$$\begin{aligned} c_1 e^{-1} + c_2 e^{-3} &= 1, \\ c_1 e^{-1} + 3c_2 e^{-3} &= 0, \end{aligned}$$

the solution of which gives $c_1 = 3e/2$ and $c_2 = -e^3/2$. From the initial conditions for solution y_2, we get the simultaneous equations

$$\begin{aligned} c_1 e^{-1} + c_2 e^{-3} &= 0, \\ c_1 e^{-1} + 3c_2 e^{-3} &= -1, \end{aligned}$$

the solution of which gives $c_1 = e/2$ and $c_2 = -e^3/2$. Hence we have

$$y_1 = \frac{3}{2} e^{-(t-1)} - \frac{1}{2} e^{-3(t-1)}, \qquad y_2 = \frac{1}{2} e^{-(t-1)} - \frac{1}{2} e^{-3(t-1)}.$$

4. Boyce and DiPrima, p. 156, Problem 27.

Solution: Let $L[y] = (1 - x\cot x)y'' - xy' + y$. For $y_1 = x$, we get $y_1' = 1$, $y_1'' = 0$. Hence we have

$$L[y_1] = 0 - x(1) + x = 0.$$

For $y_2 = \sin x$, we obtain $y_2' = \cos x$, $y_2'' = -\sin x$. Hence we have

$$L[y_2] = (1 - x\cot x)(-\sin x) - x(\cos x) + \sin x = 0.$$

Therefore both y_1 and y_2 are solutions of the given differential equation. The Wronskian of solutions y_1 and y_2 is:

$$W(y_1, y_2) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x.$$

Since $W(y_1, y_2)(\pi/2) = -1 \neq 0$, the solutions y_1 and y_2 constitute a fundamental set of solutions of the given differential equation on the interval $(0, \pi)$.

5. Boyce and DiPrima, p. 156, Problem 29.

   Solution: In standard form the given differential equation reads:

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 0.$$

   Hence $p(t) = -1 - 2/t$. Therefore the Wronskian W satisfies the differential equation

$$W' + (-1 - 2/t)W = 0,$$

   the general solution of which is $W = Ce^t t^2$, where C is an arbitrary constant.

6. Boyce and DiPrima, p. 157, Problem 33.

   Solution: We recast the given equation in standard form as:

$$y'' + \frac{p'}{p}y' + \frac{q}{p}y = 0.$$

   The Wronskian of two solutions of the preceding equation satisfies the equation

$$\frac{dW}{dt} + \frac{p'}{p}W = 0. \tag{1}$$

   An integrating factor of the equation on W is given by

$$\mu = e^{\int (p'/p)dt} = e^{\ln p} = p.$$

   Multiplying both sides of (1) by the integrating factor $\mu = p$, we obtain

$$(p(t)W(t))' = 0.$$

   Hence we have $p(t)W(t) = c$ or

$$W(t) = \frac{c}{p(t)},$$

   where c is an arbitrary constant.

7. Boyce and DiPrima, p. 157, Problem 35.

   Solution: In standard form the given differential equation reads:

$$y'' - \frac{2}{t^2}y' + \frac{3+t}{t^2}y = 0.$$

   Hence $p(t) = -2/t^2$. Therefore the Wronskian W satisfies the differential equation

$$W' - (2/t^2)W = 0,$$

   the general solution of which is $W = Ce^{-2/t}$, where C is an arbitrary constant. From $W(2) = 3$ we obtain $C = 3e$. Hence $W = 3e \cdot e^{-2/t}$ and $W(4) = 3\sqrt{e}$.

8. Boyce and DiPrima, p. 157, Problem 40.

   Solution. Suppose $t_0 \in I$ is a common point of inflection of y_1 and y_2. Then $y_1''(t_0) = y_2''(t_0) = 0$. Since $W'(t_0) = y_1(t_0)y_2''(t_0) - y_2(t_0)y_1''(t_0)$, we have $W'(t_0) = 0$. If y_1 and y_2 form a set of fundamental solutions, we must have $W(y_1, y_2)(t_0) \neq 0$. It then follows from the equation $W'(t_0) + p(t_0)W(t_0) = 0$ that $p(t_0) = 0$.

   Since y_1 and y_2 are solutions, they satisfy

$$y_1'' + p(t)y_1' + q(t)y_1 = 0, \qquad y_2'' + p(t)y_2' + q(t)y_2 = 0.$$

   Substituting $t = t_0$ in the preceding equations, we obtain

$$q(t_0)y_1(t_0) = 0, \qquad q(t_0)y_2(t_0) = 0.$$

   Since $W(y_1, y_2)(t_0) \neq 0$, $y_1(t_0)$ and $y_2(t_0)$ cannot both equal to zero. Hence we must have $q(t_0) = 0$.

9. Boyce and DiPrima, Section 3.5, p. 173, Problem 20.

   Solution: (a) The characteristic equation of the equation $y'' + 2ay' + a^2y = 0$ is $r^2 + 2ar + a^2 = (r + a)^2 = 0$, which has repeated roots $r_1 = r_2 = -a$. Hence $y_1 = e^{-at}$ is a solution of the given homogeneous equation.

   (b) By Abel's theorem, the Wronskian W of the given equation satisfies the differential equation

$$W' + p(t)W = W' + 2aW = 0,$$

   the general solution of which is $W = c_1e^{-2at}$, where c_1 is a constant. Since $W(t) = y_1y_2' - y_1'y_2$ and y_1 is determined in (a), we see that y_2 satisfies the first-order equation

$$y_1y_2' - y_1'y_2 = c_1e^{-2at}.$$

   (c) Putting $y_1 = e^{-at}$ into the differential equation for y_2 in (b), we obtain the equation

$$y_2' + ay_2 = c_1e^{-at},$$

   which has the general solution

$$y_2 = c_1te^{-at} + c_2e^{-at}.$$

   To get one solution which forms a fundamental set of solutions with y_1, we simply take $c_1 = 1$ and $c_2 = 0$, i.e., we take $y_2 = te^{-at}$.

10. Boyce and DiPrima, Section 3.5, p. 173, Problem 26.

   Solution: In standard form the given equation reads:

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 0, \qquad t > 0.$$

Putting $y = vy_1 = tv$ into the preceding equation, we obtain the equation

$$tv'' - tv' = 0.$$

Let $u = v'$. Then u satisfies the equation $u' - u = 0$, which has $u = e^t$ as a non-trivial solution. Therefore $v' = e^t$, and $v = e^t$ is a solution for v. Hence we have $y_2 = te^t$ as a second solution.

11. Boyce and DiPrima, Section 3.5, p. 173, Problem 28.

    Solution: In standard form the given equation reads:

$$y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0, \qquad x > 1.$$

Putting $y = vy_1 = e^x v$ into the preceding equation, we obtain the equation

$$e^x v'' + \left(2e^x - \frac{x}{x-1}e^x\right) v' = 0$$

or

$$v'' + \frac{x-2}{x-1}v' = 0.$$

Let $u = v'$. Then u satisfies the equation

$$u' + \left(1 - \frac{1}{x-1}\right) u = 0,$$

which has $u = (x-1)e^{-x}$ as a non-trivial solution. From the equation $v' = (x-1)e^{-x}$, we get $v = -xe^{-x}$ as one non-trivial solution. Hence $y_2 = vy_1 = -x$ is a second solution we seek. As the given homogeneous equation is linear, we may take $y_2 = x$ as the second solution.