MA 214 Calculus IV (Spring 2016)
Section 2

Homework Assignment 7
Solutions

1. Boyce and DiPrima, Section 3.5, p. 184, Problem 8.

   Solution: The given equation is: $y'' + 2y' + y = 2e^{-t}$. Let $L[y] = y'' + 2y' + y$. The characteristic equation of the homogeneous equation $L[y] = 0$ is

   $$r^2 + 2r + 1 = (r + 1)^2 = 0,$$

   which has a repeated real root $r_1 = r_2 = -1$. Using the method of undetermined coefficients to find a particular solution Y, we put $Y = At^2e^{-t}$, where the constant A is to be determined. By straightforward computations we find $L[Y] = 2Ae^{-t}$. Equating $2Ae^{-t} = 2e^{-t}$, we find $A = 1$ and thence $Y = t^2e^{-t}$. The general solution of the given equation is

   $$y = c_1e^{-t} + c_2te^{-t} + t^2e^{-t},$$

   where c_1 and c_2 are arbitrary constants.

2. Boyce and DiPrima, Section 3.5, p. 184, Problem 9.

   Solution: The given equation is: $2y'' + 3y' + y = t^2 + 3\sin t$. Let $L[y] = 2y'' + 3y' + y$. The characteristic equation of the homogeneous equation $L[y] = 0$ is

   $$2r^2 + 3r + 1 = (2r + 1)(r + 1) = 0,$$

   which has two distinct real roots $r_1 = -1/2$, $r_2 = -1$.

   We find particular solutions for the two equations $L[y] = t^2$, $L[y] = 3\sin t$ separately. Using the method of undetermined coefficients, we put $Y_1 = At^2 + Bt + C$, where the constant A, B and C are to be determined. By direct computations we find $L[Y_1] = At^2 + (6A + B)t + (4A + 3B + C)$. Hence we have $A = 1$, $6A + B = 0$, and $4A + 3B + C = 0$, which give $B = -6$ and $C = 14$. Therefore $Y_1 = t^2 - 6t + 14$.

   Next we put $Y_2 = D\cos t + E\sin t$. Then we have

   $$L[Y_2] = (3E - D)\cos t + (-3D - E)\sin t.$$

   Setting the right-hand side of the preceding equation equal to $3\sin t$, we get $3E-D = 0$, $-3D - E = 3$, which imply

   $$D = -9/10, \qquad E = -3/10.$$

Hence $Y_2 = (-9/10)\cos t - (3/10)\sin t$.

The general solution of the given equation is:

$$y = c_1 e^{-t/2} + c_2 e^{-t} + t^2 - 6t + 14 + (-9/10)\cos t - (3/10)\sin t,$$

where c_1 and c_2 are arbitrary constants.

3. Boyce and DiPrima, Section 3.5, p. 184, Problem 12.

   Solution: The given equation is: $u'' + \omega_o^2 u = \cos\omega_o t$. The characteristic equation of the accompanying homogeneous equation is $r^2 + \omega_o^2 = 0$, which has roots $r_1 = i\omega_o$, $r_2 = -i\omega_o$. Using the method of undetermined coefficients, we find a particular solution of the given equation by putting $U = t(A\cos\omega_o t + B\sin\omega_o t)$. Let $L[u] = u'' + \omega_o^2 u$. By direct computations we find $L[U] = -2\omega_o A\sin\omega_o t + 2\omega_o B\cos\omega_o t$. It follows that the coefficients A and B must satisfy $A = 0$ and $2\omega_o B = 1$, which gives $U = (t/2\omega_o)\sin\omega_o t$. Hence the general solution of the given equation is:

$$u = c_1\cos\omega_o t + c_2\sin\omega_o t + \frac{1}{2\omega_o} t\sin\omega_o t,$$

   where c_1 and c_2 are arbitrary constants.

4. Boyce and DiPrima, Section 3.5, p. 184, Problem 20.

   Solution: The given initial-value problem is:

$$y'' + 2y' + 5y = 4e^{-t}\cos 2t, \qquad y(0) = 1, \quad y'(0) = 0.$$

   Let $L[y] = y'' + 2y' + 5y$. The characteristic equation of the homogeneous equation is: $r^2 + 2r + 5 = 0$, which has roots $r_1 = -1 + 2i$, $r_2 = -1 - 2i$.

   We find a particular solution of the equation $L[y] = 4e^{-t}\cos 2t$ by the method of undetermined coefficients. We put $Y = t(Ae^{-t}\cos 2t + Be^{-t}\sin 2t)$. By direct computations we find

$$L[Y] = -4Ae^{-t}\sin 2t + 4Be^{-t}\cos 2t.$$

   Hence we have $A = 0$, $B = 1$.

   The general solution of the nonhomogeneous equation $L[y] = 4e^{-t}\cos 2t$ is

$$y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + te^{-t}\sin 2t.$$

   From the initial conditions $y(0) = 1$ and $y'(0) = 0$, we get $c_1 = 1$ and $2c_2 - c_1 = 0$, respectively. Hence $c_1 = 1$, and $c_2 = 1/2$. Therefore the solution of the given initial-value problem is:

$$y = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t + te^{-t}\sin 2t.$$

5. Boyce and DiPrima, Section 3.6, p. 184, Problem 24(a) and 25(a) .

   Solution: Problem 24(a). The characteristic equation of the homogeneous equation $y'' + 2y' + 2y = 0$ is $r^2 + 2r + 2 = 0$, which has complex conjugate roots $r_1 = -1 + i$, $r_2 = -1 - i$. By the method of undetermined coefficients, to find a particular solution of the equation
$$y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2 \sin t,$$
   we should determine a particular solution of the form
$$Y = Ae^{-t} + t\left((Bt^2 + Ct + D)e^{-t}\cos t + (Et^2 + Ft + G)e^{-t}\sin t\right),$$
   where $A, B, ..., G$ are constants to be determined.

   Problem 25(a). The characteristic equation of the homogeneous equation $y''-4y'+4y = 0$ is $r^2 - 4r + 4 = (r-2)^2 = 0$, which has the repeated real root $r_1 = r_2 = 2$. By the method of undetermined coefficients, to find a particular solution of the equation
$$y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t\sin 2t,$$
   we should seek a particular solution of the form
$$Y = At^2 + Bt + C + t^2\left((Dt + E)e^{2t}\right) + (Ft + G)\cos 2t + (Ht + I)\sin 2t,$$
   where $A, B, ..., I$ are constants to be determined.

6. Boyce and DiPrima, Section 3.6, p. 190, Problem 3.

   Solution: The characteristic equation of the homogeneous equation is $r^2 + 2r + 1 = (r+1)^2 = 0$, which has double real root $r_1 = r_2 = -1$. A fundamental set of solutions of the homogeneous equation has $y_1 = e^{-t}$, $y_2 = te^{-t}$, and the Wronskian $W(y_1, y_2) = W(e^{-t}, te^{-t}) = e^{-t}$. By the method of variation of parameters, a particular solution of the given nonhomogeneous equation is given by the formula
$$\begin{aligned} Y &= u_1y_1 + u_2y_2 \\ &= -e^{-t}\int^t \frac{3e^{-s}se^{-s}}{e^{-2s}}ds + te^{-t}\int^t \frac{3e^{-s}e^{-s}}{e^{-2s}}ds \\ &= -\frac{3}{2}t^2e^{-t} + 3t^2e^{-t} = \frac{3}{2}t^2e^{-t}. \end{aligned}$$

7. Boyce and DiPrima, Section 3.6, p. 190, Problem 6.

   Solution: The characteristic equation of the homogeneous equation is $r^2 + 9 = 0$, which has conjugate complex roots $r_1 = 3i$, $r_2 = -3i$. A fundamental set of solutions of the homogeneous equation has $y_1 = \cos 3t$, $y_2 = \sin 3t$, and the Wronskian

$W(y_1, y_2) = 3$. By the method of variation of parameters, a particular solution of the given nonhomogeneous equation is given by the formula

$$\begin{aligned}
Y &= u_1 y_1 + u_2 y_2 \\
&= -\cos 3t \int^t \frac{1}{3} 9 \sec^2 3s \sin 3s ds + \sin 3t \int^t \frac{1}{3} 9 \sec^2 3s \cos 3s ds \\
&= \cos 3t \int^{\cos 3t} \frac{1}{u^2} du + \sin 3t \int^{3t} \sec u du \\
&= -1 + \sin 3t \ln(\tan 3t + \sec 3t).
\end{aligned}$$

The general solution of the given equation is given by the formula

$$y = c_1 \cos 3t + c_2 \sin 3t - 1 + \sin 3t \ln(\tan 3t + \sec 3t).$$

8. Boyce and DiPrima, Section 3.6, p. 190, Problem 12.

   Solution: The characteristic equation of the homogeneous equation is $r^2 + 4 = 0$, which has conjugate complex roots $r_1 = 2i$, $r_2 = -2i$. A fundamental set of solutions of the homogeneous equation has $y_1 = \cos 2t$, $y_2 = \sin 2t$, and the Wronskian $W(y_1, y_2) = 2$. By the method of variation of parameters, a particular solution of the given nonhomogeneous equation is given by the formula

$$\begin{aligned}
Y &= u_1 y_1 + u_2 y_2 \\
&= -\cos 2t \int^t \frac{1}{2} g(s) \sin 2s ds + \sin 2t \int^t \frac{1}{2} g(s) \cos 2s ds \\
&= \frac{1}{2} \int^t g(s) \sin 2(t - s) ds,
\end{aligned}$$

   where we have appealed to the trigonometric formula

$$\sin 2(t - s) = \sin(2t - 2s) = \sin 2t \cos 2s - \cos 2t \sin 2s.$$

   The general solution of the given equation is given by the formula

$$y = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{2} \int^t g(s) \sin 2(t - s) ds.$$

9. Boyce and DiPrima, Section 3.6, p. 190, Problem 17.

   Solution: In standard form, the given equation reads:

$$L[y] = y'' - \frac{3}{x} y' + \frac{4}{x^2} = \ln x, \qquad x > 0.$$

It is easily verified that $L[y_1] = L[x^2] = 2 - 6 + 4 = 0$ and $L[y_2] = L[x^2 \ln x] = 3 + 2\ln x - 3(2\ln x + 1) + 4\ln x = 0$. Moreover we have

$$W(x^2, x^2 \ln x) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} = x^3 > 0$$

for $x > 0$. Hence $y_1 = x^2$ and $y_2 = x^2 \ln x$ form a fundamental set of solutions of the homogeneous equation $L[y] = 0$ on the interval $(0, \infty)$. By the method of variation of parameters, a particular solution of the given nonhomogeneous equation is given by the formula

$$\begin{aligned} Y &= u_1 y_1 + u_2 y_2 \\ &= -x^2 \int^x \frac{(\ln t)t^2 \ln t}{t^3} dt + x^2 \ln x \int^x \frac{(\ln t)t^2}{t^3} dt \\ &= -\frac{x^2}{3}(\ln x)^3 + \frac{1}{2}x^2(\ln x)^3 = \frac{1}{6}x^2(\ln x)^3. \end{aligned}$$

10. Boyce and DiPrima, Section 3.6, p. 190, Problem 19.

    Solution: In standard form, the given equation reads:

$$L[y] = y'' + \frac{x}{1-x}y' - \frac{1}{1-x} = \frac{g(x)}{1-x}, \qquad 0 < x < 1.$$

It is easily verified that $L[y_1] = L[e^x] = e^x + xe^x/(1-x) - e^x/(1-x) = 0$ and $L[y_2] = L[x] = x/(1-x) - x/(1-x) = 0$. Moreover we have

$$W(e^x, x) = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x(1-x) > 0$$

for $0 < x < 1$. Hence $y_1 = e^x$ and $y_2 = x$ form a fundamental set of solutions of the homogeneous equation $L[y] = 0$ on the interval $0 < x < 1$. By the method of variation of parameters, a particular solution of the given nonhomogeneous equation is given by the formula

$$\begin{aligned} Y &= u_1 y_1 + u_2 y_2 \\ &= -e^x \int^x \frac{g(t)t}{(1-t)^2 e^t} dt + x \int^x \frac{g(t)}{(1-t)^2} dt \\ &= \int^x \frac{xe^t - te^x}{(1-t)^2 e^t} g(t) dt. \end{aligned}$$