MA 214 Calculus IV (Spring 2016)

Section 2

Homework Assignment 8

Solutions

1. Boyce and DiPrima, Section 3.7, p. 203, Problem 3 and Problem 4.

   Solution: In both of these problems, the given expression of u is recast in the form $u = R\cos(\omega_0 t - \delta)$, where R and δ are given as follows.

   Problem 3. Here $A = R\cos\delta = 4$, $B = R\sin\delta = -2$. Hence $R = \sqrt{A^2 + B^2} = 2\sqrt{5}$. It follows that

   $$\cos\delta = \frac{2}{\sqrt{5}}, \qquad \sin\delta = -\frac{1}{\sqrt{5}}, \qquad \tan\delta = -\frac{1}{2}.$$

   From the sign of $\cos\delta$ and of $\sin\delta$, we see that the angle δ is in the 4th quadrant and $\delta = \arctan(-1/2) \approx -0.4636$ radians.

   Problem 4. Here $A = R\cos\delta = -2$, $B = R\sin\delta = -3$. Hence $R = \sqrt{A^2 + B^2} = \sqrt{13}$. It follows that

   $$\cos\delta = -\frac{2}{\sqrt{13}}, \qquad \sin\delta = -\frac{3}{\sqrt{13}}, \qquad \tan\delta = \frac{3}{2}.$$

   From the sign of $\cos\delta$ and of $\sin\delta$, we see that the angle δ is in the 3rd quadrant and $\delta = \pi + \arctan(3/2) \approx 4.1244$ radians.

2. Boyce and DiPrima, Section 3.7, p. 203, Problem 7.

   Solution: In what follows we take the acceleration due to gravity $g = 32$ ft/s^2. Let k be the spring constant in question. Then we have $k = 3\,\text{lbf}/(1/4)\,\text{ft} = 12$ lbf/ft. The mass weighs 3 lbf. Hence the initial-value problem in question is

   $$\frac{3}{32}u'' + 12u = 0, \qquad u(0) = -\frac{1}{12}, \quad u'(0) = 2.$$

   The general solution of the differential equation is

   $$u = A\cos 8\sqrt{2}t + B\sin 8\sqrt{2}t.$$

   From the initial conditions, we find $A = -1/12$, $B = \sqrt{2}/8$. Note that δ is in the 2nd quadrant and $\tan\delta = B/A = -3/\sqrt{2}$. Therefore the spring-mass system has vibration frequency $\omega_0 = 8\sqrt{2}\,\text{s}^{-1}$, period $T = \pi/(4\sqrt{2})$ s, amplitude $R = \sqrt{A^2 + B^2} = \sqrt{11/288}$ ft, and phase $\delta = \pi - \arctan(3/\sqrt{2})$.

3. Boyce and DiPrima, Section 3.7, p. 204, Problem 11.

   Solution: The spring constant of the spring is $k = 3/0.1 = 30$ N/m. The damping coefficient of the spring-mass system is $\gamma = 3/5$ N·s/m^2. Hence the displacement u of the mass is governed by the equation of motion

$$2u'' + \frac{3}{5}u' + 30u = 0 \qquad \text{or} \qquad u'' + 0.3u' + 15u = 0.$$

   The initial conditions for the motion are: $u(0) = 0.05$ m, $u'(0) = 10$ m/s. Solving the initial-value problem, we obtain

$$u = e^{-0.15t}(A\cos\mu t + B\sin\mu t),$$

   where

$$\mu = \frac{\sqrt{4km - \gamma^2}}{2m} = 3.870078 \text{ s}^{-1}, \quad A = 0.05 \text{ m}, \quad B = 0.027777 \text{ m}.$$

   The displacement u can be written in the form

$$u = 0.057201e^{-0.15t}\cos(3.870078t - 0.507087) \text{ m},$$

   and the ratio $\mu/\omega_0 = 3.870078/\sqrt{15} = 0.99925$.

4. Boyce and DiPrima, Section 6.1, p. 315, Problem 5.

   Solution: Let n be a positive integer. For $s > 0$, we have

$$\begin{aligned}
\mathcal{L}[t^n] &= \int_0^\infty t^n e^{-st}dt \\
&= \lim_{A\to\infty}\int_0^A t^n e^{-st}dt \\
&= \lim_{A\to\infty}\left(\left[\frac{t^n e^{-st}}{-s}\right]_0^A + \frac{1}{s}\int_0^A nt^{n-1}e^{-st}dt\right) \\
&= \lim_{A\to\infty}\frac{A^n e^{-sA}}{-s} + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt \\
&= \frac{n}{s}\mathcal{L}[t^{n-1}] = \frac{n(n-1)}{s^2}\mathcal{L}[t^{n-2}] = \cdots = \frac{n!}{s^n}\mathcal{L}[1] = \frac{n!}{s^{n+1}}.
\end{aligned}$$

   Putting $n = 1$ and $n = 2$, we obtain

$$\mathcal{L}[t] = \frac{1}{s^2} \qquad \text{and} \qquad \mathcal{L}[t^2] = \frac{2}{s^3},$$

   respectively.

5. Boyce and DiPrima, Section 6.1, p. 315, Problem 13.

Solution: By definition, we have

$$\mathcal{L}[e^{at}\sin bt] = \lim_{A\to\infty}\int_0^A e^{at}(\sin bt)\, e^{-st}dt = \lim_{A\to\infty}\int_0^A (\sin bt)\, e^{(a-s)t}dt.$$

Let $I = \int_0^A (\sin bt)\, e^{(a-s)t}dt$. Using integration by parts, we observe that

$$\begin{aligned} I &= \left[\frac{(\sin bt)\, e^{(a-s)t}}{a-s}\right]_0^A - \frac{b}{a-s}\int_0^A (\cos bt)\, e^{(a-s)t}dt \\ &= \frac{(\sin bA)\, e^{(a-s)A}}{a-s} - \frac{b}{a-s}\left(\left[\frac{(\cos bt)\, e^{(a-s)t}}{a-s}\right]_0^A - \int_0^A \frac{-b\sin bt}{a-s} e^{(a-s)t}dt\right). \end{aligned}$$

For $s > a$, we obtain

$$\mathcal{L}[e^{at}\sin bt] = \lim_{A\to\infty} I = \frac{b}{s-a}\left(\frac{1}{s-a} - \frac{b}{s-a}\mathcal{L}[e^{at}\sin bt]\right).$$

It follows that

$$\left(1 + \frac{b^2}{(s-a)^2}\right)\mathcal{L}[e^{at}\sin bt] = \frac{b}{(s-a)^2},$$

and

$$\mathcal{L}[e^{at}\sin bt] = \frac{b}{(s-a)^2 + b^2} \qquad \text{for} \quad s > a.$$

6. Boyce and DiPrima, Section 6.1, p. 315, Problem 16.

Solution: By definition, we have

$$\mathcal{L}[t\sin at] = \lim_{A\to\infty}\int_0^A (t\sin at)\, e^{-st}dt.$$

Let $I = \int_0^A (t\sin at)\, e^{-st}dt$. Using integration by parts, we observe that

$$I = \left[\frac{(t\sin at)\, e^{-st}}{-s}\right]_0^A + \frac{1}{s}\int_0^A (\sin at + at\cos at)\, e^{-st}dt.$$

It follows that for $s > 0$,

$$\mathcal{L}[t\sin at] = \lim_{A\to\infty} I = \frac{1}{s}\mathcal{L}[\sin at] + \frac{a}{s}\mathcal{L}[t\cos at].$$

On the other hand, a similar calculation shows that for $s > 0$

$$\begin{aligned} \mathcal{L}[t\cos at] &= \lim_{A\to\infty}\left(\left[\frac{(t\cos at)\, e^{-st}}{-s}\right]_0^A + \frac{1}{s}\int_0^A (\cos at - at\sin at)\, e^{-st}dt\right) \\ &= -\frac{a}{s}\mathcal{L}[t\sin at] + \frac{1}{s}\mathcal{L}[\cos at]. \end{aligned}$$

Hence we have

$$\mathcal{L}[t \sin at] = \frac{1}{s}\mathcal{L}[\sin at] - \frac{a^2}{s^2}\mathcal{L}[t \sin at] + \frac{a}{s^2}\mathcal{L}[\cos at].$$

It follows that

$$\left(1 + \frac{a^2}{s^2}\right)\mathcal{L}[t \sin at] = \frac{1}{s} \cdot \frac{a}{s^2 + a^2} + \frac{a}{s^2} \cdot \frac{s}{s^2 + a^2},$$

or

$$\mathcal{L}[t \sin at] = \frac{2as}{(s^2 + a^2)^2} \qquad \text{for} \quad s > 0.$$

7. Boyce and DiPrima, Section 6.2, p. 324, Problem 5.

   Solution: We have

$$\mathcal{L}^{-1}\left[\frac{2s + 2}{s^2 + 2s + 5}\right] = 2\mathcal{L}^{-1}\left[\frac{s + 1}{(s + 1)^2 + 2^2}\right] = 2e^{-t}\cos 2t.$$

8. Boyce and DiPrima, Section 6.2, p. 324, Problem 8.

   Solution: First we break up $F(s)$ into partial fractions:

$$\frac{8s^2 - 4s + 12}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4}.$$

   Multiplying both sides of the preceding equation by $s(s^2 + 4)$, we obtain

$$8s^2 - 4s + 12 = A(s^2 + 4) + (Bs + C)s = (A + B)s^2 + Cs + 4A,$$

   which implies $A = 3$, $B = 5$, and $C = -4$. It follows that

$$\mathcal{L}^{-1}\left[\frac{8s^2 - 4s + 12}{s(s^2 + 4)}\right] = \mathcal{L}^{-1}\left[\frac{3}{s}\right] + \mathcal{L}^{-1}\left[\frac{5s - 4}{s^2 + 4}\right] = 3 + 5\cos 2t - 2\sin 2t.$$

9. Boyce and DiPrima, Section 6.2, p. 325, Problem 13.

   Solution: Finding the Laplace transform of both sides of the given equation, we have

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + 2Y(s) = 0.$$

   Using the initial conditions $y(0) = 0$ and $y'(0) = 1$, we solve the preceding equation for $Y(s)$ and obtain

$$Y(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s - 1)^2 + 1^2}.$$

   Hence $y(t) = e^t \sin t$ is the solution of the given initial-value problem.

10. Boyce and DiPrima, Section 6.2, p. 325, Problem 17.

Solution: Finding the Laplace transform of both sides of the given equation and appealing to the initial conditions, we obtain the following equation for $Y(s)$:

$$(s^4 - 4s^3 + 6s^2 - 4s + 1)Y(s) = s^2 - 4s + 7.$$

Hence we have

$$\begin{aligned} Y(s) = \frac{s^2 - 4s + 7}{(s-1)^4} &= \frac{(s-1)^2 - 2(s-1) + 4}{(s-1)^4} \\ &= \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4}. \end{aligned}$$

Therefore the solution to the given initial-value problem is:

$$y(t) = te^t - t^2e^t + \frac{2}{3}t^3e^t.$$

11. Boyce and DiPrima, Section 6.2, p. 325, Problem 22.

Solution: Finding the Laplace transform of both sides of the given equation and appealing to the initial conditions, we obtain the following equation for $Y(s)$:

$$(s^2 - 2s + 2)Y(s) = \frac{1}{s+1} + 1,$$

or

$$Y(s) = \frac{1}{(s+1)(s^2 - 2s + 2)} + \frac{1}{s^2 - 2s + 2}.$$

Using partial fractions, we recast $Y(s)$ as

$$\begin{aligned} Y(s) &= \frac{1}{5}\left(\frac{1}{s+1} - \frac{s-8}{s^2 - 2s + 2}\right) \\ &= \frac{1}{5}\left(\frac{1}{s+1} - \frac{s-8}{(s-1)^2 + 1^2}\right) = \frac{1}{5}\left(\frac{1}{s+1} - \frac{s-1}{(s-1)^2 + 1^2} - \frac{7}{(s-1)^2 + 1^2}\right). \end{aligned}$$

Hence the solution of the given initial-value problem is:

$$y(t) = \frac{1}{5}\left(e^{-t} - e^t \cos t + 7e^t \sin t\right).$$