MA 214 Calculus IV (Spring 2016)

Section 2

Homework Assignment 9

Solutions

In what follows the Heaviside function, written as $u_c(t)$ in the text of Boyce and Diprima, is denoted by $H(t-c)$.

1. Boyce and DiPrima, Section 6.3, p. 333, Problem 6.

   Solution: We recast the given function as

$$f(t) = (t-1)\big(H(t-1) - H(t-2)\big) + (3-t)\big(H(t-2) - H(t-3)\big).$$

   A sketch of the graph of f is depicted in Figure 1.

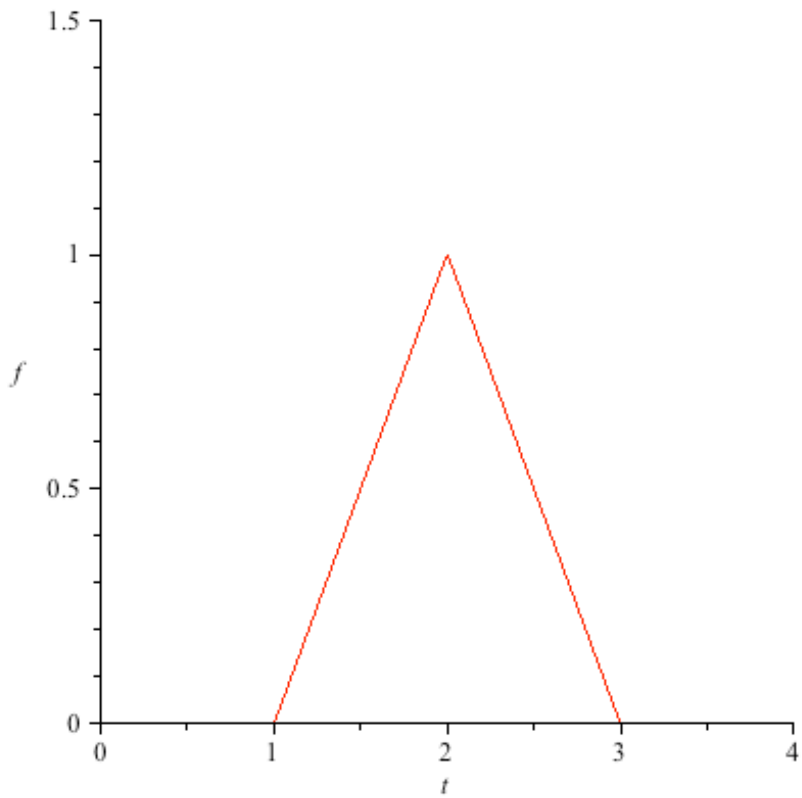

Figure 1: Sketch of graph of f.

2. Boyce and DiPrima, Section 6.3, p. 333, Problem 11.

   Solution: We recast the given function as

$$f(t) = (t-2)H(t-2) - H(t-2) - (t-3)H(t-3) - H(t-3).$$

   Using formulas 12 and 13 in the Table of Laplace Transforms and the fact that $\mathcal{L}\{t\} = 1/s^2$ (cf. formula 3 in the Table), we obtain

$$\mathcal{L}\{f(t)\} = \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}.$$

3. Boyce and DiPrima, Section 6.3, p. 333, Problem 15 and Problem 17.

   Solution: Problem 15. First we recast the given function in terms of the Heaviside function:

$$\begin{aligned} f(t) &= (t-\pi)[H(t-\pi) - H(t-2\pi)] \\ &= (t-\pi)H(t-\pi) - (t-2\pi)H(t-2\pi) - \pi H(t-2\pi). \end{aligned}$$

   Hence we have

$$\mathcal{L}[f] = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s}.$$

   Problem 17. Since

$$\begin{aligned} f(t) &= (t-3)H(t-2) - (t-2)H(t-3) \\ &= (t-2)H(t-2) - H(t-2) - (t-3)H(t-3) - H(t-3), \end{aligned}$$

   we have

$$\mathcal{L}[f] = \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}.$$

4. Boyce and DiPrima, Section 6.3, p. 333, Problem 23.

   It is easy to use partial fractions to get

$$\frac{s-2}{s^2-4s+3} = \frac{s-2}{(s-1)(s-3)} = \frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s-3}\right).$$

   Hence we have

$$\begin{aligned} \mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}\left[\frac{e^{-s}}{2}\left(\frac{1}{s-1} + \frac{1}{s-3}\right)\right] \\ &= \frac{1}{2}H(t-1)\left(e^{t-1} + e^{3(t-1)}\right). \end{aligned}$$

5. Boyce and DiPrima, Section 6.3, p. 335, Problem 37.

   Solution: The given function is periodic with period $T = 1$. Using the formula given in Problem 34 of the text of Boyce and DiPrima (which was derived in class), we have

$$\begin{aligned}
\mathcal{L}[f(t)] &= \frac{1}{1-e^{-s}} \int_0^1 te^{-st}dt \\
&= \frac{1}{1-e^{-s}} \left(\left[\frac{te^{-st}}{-s} \right]_0^1 + \frac{1}{s} \int_0^1 e^{-st}dt \right) \\
&= \frac{1}{1-e^{-s}} \left(\frac{e^{-s}}{-s} + \frac{1}{s} \left[\frac{e^{-st}}{-s} \right]_0^1 \right) \\
&= \frac{1-e^{-s}-se^{-s}}{s^2(1-e^{-s})}.
\end{aligned}$$

For problems 6 to 9, I will write down the equations with $u_c(t)$ replaced by $H(t-c)$.

6. $y'' + 4y = \sin t - \sin(t-2\pi)H(t-2\pi), \qquad y(0) = 0, \quad y'(0) = 0.$

   Solution: Let $Y(s) = \mathcal{L}[y(t)]$, the Laplace transform of the solution $y(t)$. Taking the Laplace transform of both sides of the given differential equation, we appeal to the initial conditions to obtain

$$(s^2+4)Y(s) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1},$$

   which implies

$$\begin{aligned}
Y(s) &= \frac{1}{(s^2+1)(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)} \\
&= \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) - \frac{e^{-2\pi s}}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right).
\end{aligned}$$

   Hence we have

$$y(t) = \frac{1}{6}\left(2\sin t - \sin 2t - [2\sin(t-2\pi) - \sin 2(t-2\pi)]H(t-2\pi)\right).$$

7. $y'' + 3y' + 2y = H(t-2), \qquad y(0) = 0, \quad y'(0) = 1.$

   Solution: Taking the Laplace transform of both sides of the given equation, we obtain

$$s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \frac{e^{-2s}}{s}.$$

Substituting $y(0) = 0$ and $y'(0) = 1$, we solve the preceding equation for $Y(s)$ and obtain

$$\begin{aligned} Y(s) &= \frac{e^{-2s}}{s(s^2+3s+2)} + \frac{1}{s^2+3s+2} \\ &= \frac{e^{-2s}}{s(s+1)(s+2)} + \frac{1}{(s+1)(s+2)} \\ &= e^{-2s}\left(\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right) + \frac{1}{s+1} - \frac{1}{s+2}. \end{aligned}$$

Hence the solution of the given initial-value problem is

$$y(t) = H(t-2)\left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right] + e^{-t} - e^{-2t}.$$

8. $y'' + y = g(t), \qquad y(0) = (0), \quad y'(0) = 1, \quad g(t) = \begin{cases} t/2, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}.$

Solution: First we recast $g(t)$ in terms of the Heaviside function:

$$\begin{aligned} g(t) &= \frac{t}{2}(1 - H(t-6)) + 3H(t-6) \\ &= \frac{t}{2} - \frac{t-6}{2}H(t-6) - 3H(t-6) + 3H(t-6) \\ &= \frac{t}{2} - \frac{t-6}{2}H(t-6). \end{aligned}$$

Taking the Laplace transform of both sides of the given equation, we have

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{2s^2} - \frac{e^{-\pi s}}{2s^2},$$

which implies

$$\begin{aligned} Y(s) &= \frac{1}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2(s^2+1)} - \frac{1}{2}\cdot\frac{e^{-6s}}{s^2(s^2+1)} \\ &= \frac{1}{s^2+1} + \frac{1}{2}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) - \frac{e^{-6s}}{2}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right). \end{aligned}$$

Hence the solution of the given initial-value problem is:

$$\begin{aligned} y(t) &= \sin t + \frac{1}{2}(t - \sin t) - \frac{1}{2}[(t-6) - \sin(t-6)]H(t-6) \\ &= \frac{1}{2}(t + \sin t) - \frac{1}{2}[(t-6) - \sin(t-6)]H(t-6). \end{aligned}$$

9. $y^{(4)} + 5y'' + 4y = 1 - H(t-\pi), \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y^{(3)}(0) = 0.$

Solution: Taking the Laplace transform of both sides of the given equation and then solving for $Y(s)$, we find

$$\begin{aligned} Y(s) &= \frac{1 - e^{-\pi s}}{s(s^2+1)(s^2+4)} \\ &= (1 - e^{-\pi s}) \left(\frac{1}{4s} - \frac{s}{3(s^2+1)} + \frac{s}{12(s^2+4)} \right), \end{aligned}$$

where we have used the factorization $s^4 + 5s^2 + 4 = (s^2+1)(s^2+4)$ and the partial-fraction decomposition

$$\frac{1}{s(s^2+1)(s^2+4)} = \frac{1}{4s} - \frac{s}{3(s^2+1)} + \frac{s}{12(s^2+4)}.$$

Hence the solution of the given initial-value problem is:

$$y(t) = \frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t + \left(\frac{1}{4} - \frac{1}{3}\cos(t-\pi) + \frac{1}{12}\cos 2(t-\pi) \right) H(t-\pi).$$

10. Boyce and DiPrima, Section 6.4, p. 341, Problem 15.

Solution: The function g in question is defined as follows:

$$g(t) = \begin{cases} 0, & 0 \le t < t_o \\ h(t-t_o)/k, & t_o \le t < t_o + k \\ -h(t-t_o-2k)/k, & t_o + k \le t < t_o + 2k \\ 0, & t_o + 2k \le t < \infty. \end{cases}$$

In terms of the Heaviside function, we have

$$\begin{aligned} g(t) = & \frac{h}{k}(t-t_o)[H(t-t_o) - H(t-t_o-k)] \\ & \qquad + \frac{-h}{k}(t-t_o-2k)[H(t-t_o-k) - H(t-t_o-2k)] \\ = & \frac{h}{k}\left[(t-t_o)H(t-t_o) - 2(t-t_o-k)H(t-t_o-k) + (t-t_o-2k)H(t-t_o-2k)\right]. \end{aligned}$$