MA 214 002 Calculus IV

Quiz 10

Solutions

1. (50%) Solve the initial value problem

$$y'' - 2y' + 5y = 2\delta(t-5), \qquad y(0) = 1, \quad y'(0) = 0.$$

Solution: Taking the Laplace transform of both sides of the equation, applying the initial conditions, and simplifying, we obtain

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + 5Y(s) = 2e^{-5s},$$

or

$$(s^2 - 2s + 5)Y(s) = s - 2 + 2e^{-5s}.$$

Thus we have

$$\begin{aligned} Y(s) &= \frac{s-2}{s^2-2s+5} + \frac{2}{s^2-2s+5} \cdot e^{-5s} \\ &= \frac{s-1}{(s-1)^2+2^2} - \frac{1}{(s-1)^2+2^2} + \frac{2}{(s-1)^2+2^2} \cdot e^{-5s} \\ &= \frac{s-1}{(s-1)^2+2^2} - \frac{1}{2} \cdot \frac{2}{(s-1)^2+2^2} + \frac{2}{(s-1)^2+2^2} \cdot e^{-5s}, \end{aligned}$$

which implies that the solution of the given initial-value problem is given by

$$y(t) = e^t \cos 2t - \frac{1}{2}e^t \sin 2t + e^{t-5} \sin 2(t-5)\, H(t-5).$$

2. (20%) Express the inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{1}{s+1} \cdot \frac{1}{s+2}\right\}$ as a convolution integral and evaluate the integral.

Solution: Since $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$ and $\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$, we have

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s+1} \cdot \frac{1}{s+2}\right\} &= e^{-t} * e^{-2t} = \int_0^t e^{-(t-\tau)}e^{-2\tau}\, d\tau \\ &= e^{-t} \int_0^t e^{-\tau}\, d\tau = e^{-t}(1 - e^{-t}) = e^{-t} - e^{-2t}. \end{aligned}$$

3. (30%) Use the Laplace transform to solve the following integral equation:

$$y(t) - \int_0^t \sin(t-\tau)y(\tau)\,d\tau = 1.$$

Solution: Taking the Laplace transform of both sides of the given equation, we obtain

$$Y(s) - \frac{1}{s^2+1} \cdot Y(s) = \frac{1}{s}.$$

Solving the preceding equation for $Y(s)$, we get

$$Y(s) = \frac{1}{s(1-\frac{1}{s^2+1})} = \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3} = \frac{1}{s} + \frac{1}{2!} \cdot \frac{2!}{s^3}.$$

Therefore the solution of the given integral equation is:

$$y(t) = 1 + \frac{1}{2}t^2.$$