MA 214 002 Calculus IV

Quiz 1 and Quiz 2
Solutions

1. (a) (25%) Solve the initial-value problem

$$ty' + 2y = t^3, \qquad y(1) = 1, \qquad t > 0.$$

Solution: In standard form the given linear differential equation reads:

$$y' + \frac{2}{t}y = t^2, \qquad t > 0.$$

An integrating factor of the given equation is

$$\mu(t) = \exp\left(\int \frac{2}{t}dt\right) = \exp(2\ln t) = \exp(\ln(t^2)) = t^2.$$

Multiplying both sides of the equation by the integrating factor, we obtain

$$(t^2y)' = t^4,$$

which yields

$$t^2y = \frac{t^5}{5} + C \qquad \text{or} \qquad y = \frac{t^3}{5} + \frac{C}{t^2},$$

where C is a constant. From the initial condition $y(1) = 1$, we determine that

$$C = 1 - \frac{1}{5} = \frac{4}{5}.$$

Hence the solution of the given initial-value problem is

$$y = \frac{t^3}{5} + \frac{4}{5t^2}.$$

(b) (25%) Find in explicit form the solution of the initial-value problem

$$\frac{dy}{dx} = xy^3, \qquad y(0) = -1.$$

Determine the interval in which the solution is defined.

Solution: The given first-order equation is separable. Separating the variables and integrating, we have

$$\int y^{-3}\,dy = \int x\,dx \qquad \text{or} \qquad -\frac{1}{2y^2} = \frac{x^2}{2} + C,$$

where C is a constant. From the initial condition $y(0) = -1$, we get

$$C = -1/2.$$

Hence the solution is given implicitly by the equation

$$\frac{1}{y^2} = 1 - x^2,$$

from which we obtain the explicit solution

$$y = -\frac{1}{\sqrt{1-x^2}}.$$

The interval of definition of the solution is $-1 < x < 1$.

2. (50%) At time $t = 0$, a 200-gal tank contains 10 lb of salt dissolved in 100 gal of water. Salt solution of concentration 2 lb/gal flows in the tank at the rate of 2 gal/min. For each of the following situations, formulate (but DO NOT SOLVE) the initial-value problem that governs the mass $M(t)$ (in lb) of salt in the tank at time t for $0 \le t < t_f$, where $[0, t_f)$ is the longest half-open interval on which the solution to the initial-value problem in question has physical meaning (e.g., t_f is the instant at which the tank becomes empty or begins to overflow; $t_f = \infty$, etc.); specify t_f in your formulation.

   (a) The well-mixed solution leaves the tank at 3 gal/min.

   (b) The well-mixed solution leaves the tank at 1 gal/min.

   (c) The well-mixed solution leaves the tank at 1 gal/min, and water evaporates from the tank at 1 gal/min.

   Solution: For all three cases, the initial condition is $M(0) = 10$ (lb).

   (a) The differential equation that governs $M(t)$ is:

$$\frac{dM}{dt} = 4 - \frac{3M}{100-t};$$

$t_f = 100$ (min).

(b) The differential equation that governs $M(t)$ is:

$$\frac{dM}{dt} = 4 - \frac{M}{100+t};$$

$t_f = 100$ (min).

(c) The differential equation that governs $M(t)$ is:

$$\frac{dM}{dt} = 4 - \frac{M}{100};$$

$t_f = +\infty$.