MA 214 002 Calculus IV

Quiz 3 Solutions

1. Consider the solutions of the differential equation $y' = y^2(y-3)$.

   (a) (20%) Determine the equilibrium solutions of the given equation.

   (b) (30%) Draw the phase line, and classify each equilibrium solution as to whether it is asymptotically stable, unstable, or semistable.

   (c) (30%) Sketch several graphs of solutions $y = y(t)$ in the ty-plane for $t \geq 0$ with $y(0) \in (-\infty, \infty)$. Your graphs should include a representative for each class of solutions as defined by qualitative features such as increasing, decreasing, and concavity.

Solution: (a) Let $f(y) = y^2(y-3)$. The equilibrium solutions of the given differential equation are given by the critical points of $f(y)$, i.e., the roots of $f(y) = 0$, which are clearly: $y = 0$, and $y = 3$.

(b) The critical points divide $(-\infty, \infty)$ into three open intervals, namely: $(-\infty, 0)$, $(0, 3)$, and $(3, \infty)$. It is easy to see that $y' = f(y) < 0$ and $y(t)$ is decreasing for $y \in (-\infty, 0)$ and $y \in (0, 3)$; $y' = f(y) > 0$ and $y(t)$ is increasing for $y \in (3, \infty)$. The phase line of the system modeled by the given differential equation is shown in Figure 1. It follows that the critical points $y = 0$ and $y = 3$ are semistable and unstable, respectively.

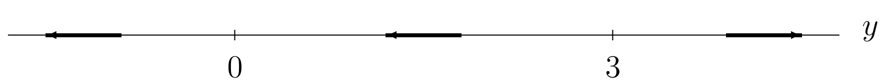

Figure 1: Phase line of system modeled by $y' = y^2(y-3)$, $-\infty < y_0 < \infty$.

(c) To determine the concavity of solution curves, we examine the sign of $y'' = f(y)f'(y)$. By direct differentiation, we find $f'(y) = 3y^2 - 6y$ and f assumes a local maximum at $y = 0$ and a local minimum at $y = 2$. Thus $f'(y) > 0$ on the intervals $(-\infty, 0)$ and $(2, \infty)$; $f'(y) < 0$ on the interval $(0, 2)$. From the sign of $f(y)$ and of $f'(y)$, we infer the concavity of solution curves on various intervals for y; see Table 1.

Intervals for y	$(-\infty, 0)$	$(0, 2)$	$(2, 3)$	$(3, \infty)$
$y' = f(y)$	$-$	$-$	$-$	$+$
$f'(y)$	$+$	$-$	$+$	$+$
$y'' = f(y)f'(y)$	$-$	$+$	$-$	$+$
Concavity	CD	CU	CD	CU

Table 1: Concavity of solution curves on various intervals for y.

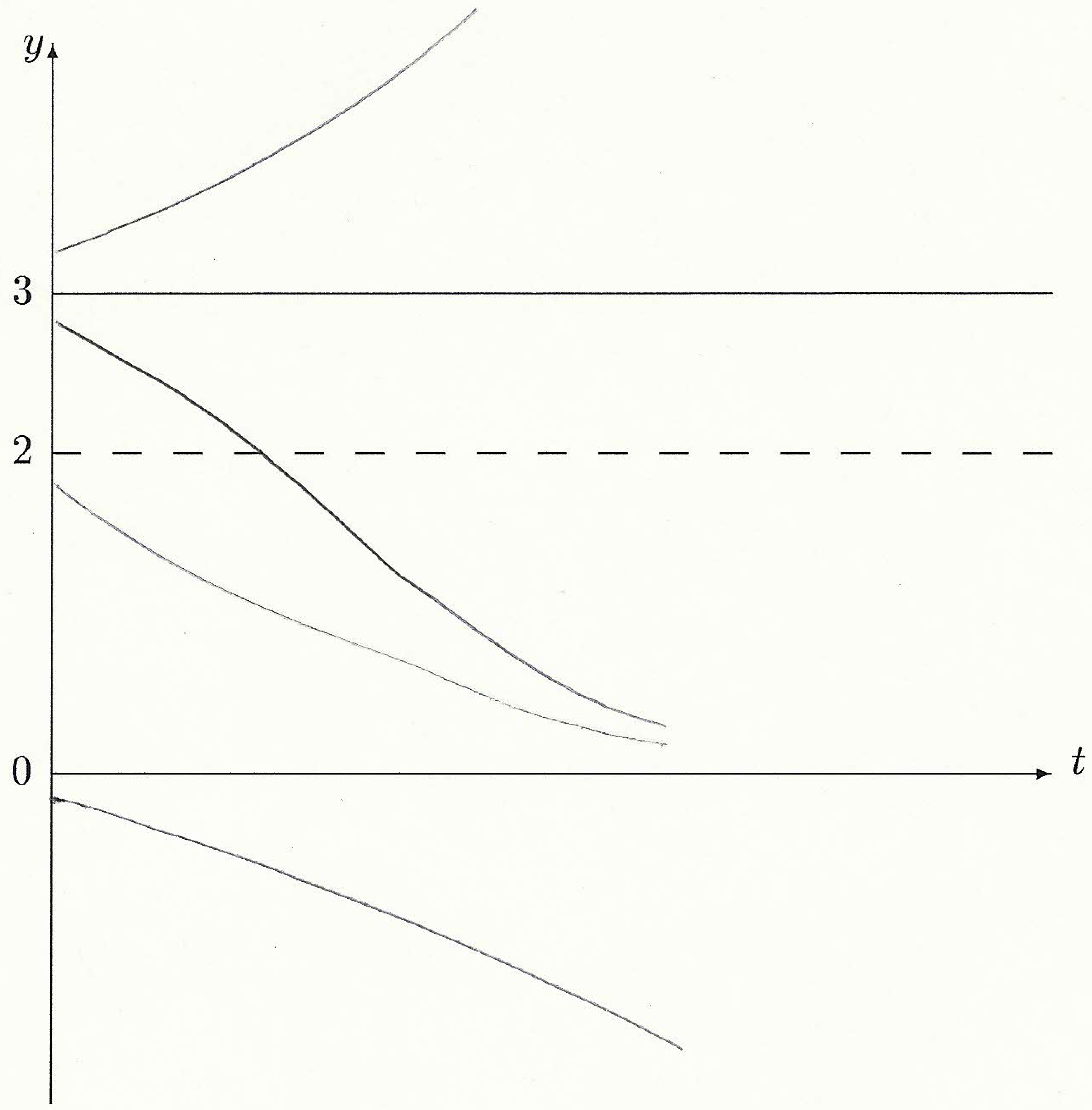

Figure 2: Representative solution curves of $y' = y^2(y-3)$.

Four representative solution curves are sketched in Figure 2.

2. For both problems below, no credit will be given unless you justify your answer.

(a) (10%) Give the interval of t on which the unique solution $y(t)$ of the initial-value problem

$$(t^2-1)y' + ty = \sin t, \qquad y(2) = 1$$

is defined.

Solution: In standard form, the given equation reads: $y' + p(t)y = g(t)$, where

$$p(t) = \frac{t}{t^2-1}, \qquad g(t) = \frac{\sin t}{t^2-1}.$$

Since both $p(t)$ and $g(t)$ are continuous on $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$ and $t_0 = 2 \in (1,\infty)$, by Theorem 2.4.1 of Boyce and DiPrima's text the unique solution $y(t)$ of the given initial-value problem is defined for each $t \in (1,\infty)$.

(b) (10%) Give the region of points (t_0, y_0) in the ty-plane where Theorem 2.4.2 of Boyce and DiPrima's text guarantees the existence of a unique solution $y(t)$ of the equation

$$y' = \frac{ty^2}{1+t^2}$$

that satisfies the initial condition $y(t_0) = y_0$.

Solution: Let $f(t,y) = ty^2/(1+t^4)$. Since f and $\partial f/\partial y = 2ty/(1+t^4)$ are continuous at every point (t,y) in the ty-plane, the required region is the entire ty-plane.