MA 214 002 Calculus IV (Spring 2016)

Solutions to Quiz 4 and Quiz 5

1. (a) (25%) Show that the differential equation

$$\cos x \cos y \, dx - 2 \sin x \sin y \, dy = 0$$

is not exact, but becomes exact when multiplied by the integrating factor $\mu = \cos y$.

(b) (25%) Solve the given equation.

Solution: Let $M = \cos x \cos y$, and $N = -2 \sin x \sin y$. Then

$$\frac{\partial M}{\partial y} = -\cos x \sin y, \qquad \frac{\partial N}{\partial x} = -2 \cos x \sin y.$$

Since $\partial M/\partial y \neq \partial N/\partial x$, the given equation is not exact.

After multiplying the equation by $\mu = \cos y$, the equation becomes

$$\cos x \cos^2 y \, dx - 2 \sin x \sin y \cos y \, dy = 0.$$

Let $\widetilde{M} = \cos x \cos^2 y$, and $\widetilde{N} = -2 \sin x \sin y \cos y$. Since

$$\frac{\partial \widetilde{M}}{\partial y} = -2 \cos x \cos y \sin y = \frac{\partial \widetilde{N}}{\partial x},$$

the equation has become exact.

We seek a function $\psi = \psi(x, y)$ such that $\partial\psi/\partial x = \widetilde{M}$ and $\partial\psi/\partial y = \widetilde{N}$. Integrating both sides of the equation $\partial\psi/\partial x = \cos x \cos^2 y$ with respect to x while keeping y fixed, we obtain

$$\psi(x, y) = \sin x \cos^2 y + f(y), \tag{1}$$

where f is a differentiable function of y. Taking the partial derivative of both sides of (1) with respect to y, we have

$$\frac{\partial \psi}{\partial y} = -2 \sin x \cos y \sin y + \frac{df}{dy}.$$

Since $\partial\psi/\partial y = \widetilde{N} = -2 \sin x \sin y \cos y$, we conclude that $df/dy = 0$. Without loss of generality, we take $f(y) = 0$ and $\psi(x, y) = \sin x \cos^2 y$. The solutions of the given differential equation are given implicitly by the equation

$$\psi(x, y) = \sin x \cos^2 y = C,$$

where C is a constant.

2. (a) (30%) Find the general solution of the following differential equations:

   (i) $2y'' - 3y' - 2 = 0$.

   (ii) $y'' + 6y' + 13y = 0$.

   (b) (20%) Solve the following initial-value problem:

$$4y'' + 4y' + y = 0, \qquad y(0) = 2, \quad y'(0) = 1.$$

Solution: (a) (i) The characteristic equation of the given differential equation is

$$2r^2 - 3r - 2 = (2r+1)(r-2) = 0,$$

the roots of which are: $r_1 = -1/2$, $r_2 = 2$. Hence the general solution of the given differential equation is

$$y = c_1 e^{-t/2} + c_2 e^{2t}, \qquad \text{where } c_1 \text{ and } c_2 \text{ are constants.}$$

(ii) The characteristic equation of the given differential equation is $r^2 + 6r + 13 = 0$. By the quadratic formula, the roots of the characteristic equation are found to be

$$r_1 = -3 + 2i, \qquad r_2 = -3 - 2i.$$

Hence the general solution of the given differential equation is

$$y = c_1 e^{-3t} \cos 2t + c_2 e^{-3t} \sin 2t, \qquad \text{where } c_1 \text{ and } c_2 \text{ are constants.}$$

(b) The characteristic equation of the given differential equation

$$4r^2 + 4r + 1 = (2r+1)^2 = 0,$$

which has a double real root $r_1 = r_2 = -1/2$. Hence the general solution of the given differential equation is

$$y(t) = c_1 e^{-t/2} + c_2 t\, e^{-t/2}, \qquad \text{where } c_1 \text{ and } c_2 \text{ are constants.} \tag{2}$$

It follows from (2) that

$$y'(t) = -\frac{1}{2} c_1 e^{-t/2} + c_2 e^{-t/2} + -\frac{1}{2} c_2 t\, e^{-t/2}. \tag{3}$$

Substituting $t = 0$ in (2) and (3), we obtain the simultaneous linear equations

$$\begin{aligned} c_1 &= 2, \\ -\frac{1}{2} c_1 + c_2 &= 1, \end{aligned}$$

from which we find $c_1 = 2$ and $c_2 = 2$. Hence the solution to the given initial-value problem is

$$y(t) = 2e^{-t/2} + 2t\, e^{-t}/2 = 2e^{-t/2}(1+t).$$