MA 214 002 Calculus IV (Spring 2016)

Solutions to Quiz 6

1. Consider the differential equation

$$x^2y'' - x(x+2)y' + (x+2)y = 0, \qquad \text{where } 0 < x < \infty.$$

   (a) (20%) Verify that $y_1(x) = x$ and $y_2(x) = xe^x$ are solutions of the given differential equation.

   (b) (20%) Prove that y_1 and y_2 constitute a fundamental set of solutions of the given differential equation on the interval $(0, \infty)$.

   (c) (10%) Write down the general solution of the given equation.

Solution: (a) Let $L(y) = x^2y'' - x(x+2)y' + (x+2)y$, Then

$$\begin{aligned} L(y_1) &= L(x) = -x(x+2) + (x+2)x = 0, \\ L(y_2) &= L(xe^x) = x^2(2e^x + xe^x) - x(x+2)(e^x + xe^x) + (x+2)xe^x \\ &= x^2e^x(x+2) - xe^x(x+2) - x^2e^x(x+2) + (x+2)xe^x = 0. \end{aligned}$$

Hence y_1 and y_2 are solutions of the equation $L(y) = 0$.

(b) It suffices to show that the Wronskian $W(y_1, y_2)(x) \neq 0$ for some $x = x_0$ in $(0, \infty)$. By direct computations, we have

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = x^2e^x > 0 \qquad \text{for each } x \in (0, \infty).$$

Therefore y_1 and y_2 constitute a fundamental set of solutions of the given homogeneous equation.

(c) The general solution of the given equation is:

$$y = c_1x + c_2xe^x,$$

where c_1 and c_2 are arbitrary constants.

2. (50%) Given that $y_1(t) = t$ is a solution of the equation

$$t^2 y'' - ty' + y = 0, \qquad (0 < t < \infty)$$

use the method of reduction of order to find a second independent solution y_2.

Solution: In standard form, the given equation reads:

$$y'' - \frac{1}{t} y' + \frac{1}{t^2} y = 0,$$

which has $p(t) = -1/t$. We seek a solution of the form $y = y_1(t)v(t) = tv$, where v satisfies the equation $tv'' + (2y_1' + py_1)v' = 0$. For $y_1 = t$, we see that v satisfies the equation

$$tv'' + \left(2 - \frac{1}{t} \cdot t\right) v' = 0, \qquad \text{or} \qquad tv'' + v' = 0.$$

Let $u = v'$. Then u satisfies the equation

$$u' + \frac{1}{t} u = 0. \tag{1}$$

An integrating factor of equation (1) is:

$$\mu = e^{\int (1/t) dt} = e^{\ln t} = t.$$

Multiplying both sides of (1) by $\mu = t$, we get

$$(ut)' = 0, \qquad \text{or} \qquad u = C/t,$$

where C is an arbitrary constant. As we need only one solution, we take $C = 1$. As $v' = u = t$, we have $v = \ln t$. Therefore a second solution of the given equation is:

$$y_2 = y_1 v = t \ln t.$$