MA 214 002 Calculus IV (Spring 2016)

Solutions to Quiz 9

1. (10%) Express the function $f(t) = \begin{cases} \sin t & \text{for } 0 \le t < \pi \\ 0 & \text{for } \pi \le t < 2\pi \\ \sin t & \text{for } 2\pi \le t < 3\pi \\ 0 & \text{for } 3\pi \le t < \infty \end{cases}$ in terms of the Heaviside function.

   Solution: $f(t) = \sin t\big(H(t) - H(t-\pi)\big) + \sin t\big(H(t-2\pi) - H(t-3\pi)\big)$.

2. (20%) Find the Laplace transform of the function $f(t) = (t^2 - t)H(t-1)$.

   Solution: Let $\tau = t - 1$. Then $t = \tau + 1$. Then we have

$$\begin{aligned} f(t) &= t(t-1)H(t-1) = (\tau+1)\tau H(\tau) \\ &= \tau^2 H(\tau) + \tau H(\tau) = (t-1)^2 H(t-1) + (t-1)H(t-1). \end{aligned}$$

   Hence the required Laplace transform is

$$F(s) = \mathcal{L}^{-1}\{f(t)\} = \frac{2}{s^3} \cdot e^{-s} + \frac{1}{s^2} \cdot e^{-s}.$$

3. (20%) Find the inverse Laplace transform of the function $F(s) = \dfrac{e^{-s}}{(s+2)^3}$.

   Solution: Note that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2} \cdot \frac{2!}{(s+2)^3}\right\} = \frac{1}{2}t^2 e^{-2t}.$$

   Hence we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+2)^3}\right\} = \frac{1}{2}(t-1)^2 e^{-2(t-1)} H(t-1).$$

4. (50%) Use the Laplace transform to solve the initial-value problem

$$y'' + 4y = H(t-1) - H(t-2), \qquad y(0) = 0, \quad y'(0) = 1.$$

   Solution: Let $\mathcal{L}\{y(t)\} = Y(s)$. Taking the Laplace transform of both sides of the given equation, we have

$$s^2 Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$

Using the initial conditions and simplifying, we get

$$(s^2+4)Y(s) = 1 + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \qquad \text{or} \qquad Y(s) = \frac{1}{s^2+4} + \frac{1}{s(s^2+4)}\left(e^{-s} - e^{-2s}\right).$$

To proceed further, we recast the expression $1/[s(s^2+4)]$ by means of partial fractions. We put

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}.$$

Multiplying both sides of the preceding equation by $s(s^2+4)$, we obtain

$$1 = A(s^2+4) + (Bs+C)s.$$

Putting $s = 0$, we get $A = 1/4$. Thus we have

$$1 = \frac{1}{4}s^2 + 1 + Bs^2 + Cs \qquad \text{or} \qquad 0 = (B + \frac{1}{4})s^2 + Cs.$$

Comparing coefficients of the last equation, we obtain $B = -1/4$, $C = 0$. Hence we conclude that

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right).$$

We recast $Y(s)$ as

$$Y(s) = \frac{1}{2}\cdot\frac{2}{s^2+2^2} + \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+2^2}\right)\cdot e^{-s} - \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+2^2}\right)\cdot e^{-2s}.$$

Hence the required solution is

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}\big(1 - \cos 2(t-1)\big)H(t-1) - \frac{1}{4}\big(1 - \cos 2(t-2)\big)H(t-2).$$