MA 214 002 Calculus IV (Spring 2016)

Answers to Review Problems for Final Exam (B)

1. (a) Let $P(x,y) = x^3 + y/x$ and $Q(x,y) = y^2 + \ln x$. The given first-order equation is exact because $\partial P/\partial y = \partial Q/\partial x = 1/x$. The solution of the given initial value problem is:
$$\frac{x^4}{4} + \frac{y^3}{3} + y\ln x = \frac{35}{12}.$$
(b) The given equation is linear. The solution is:
$$y(t) = \frac{\pi^3 - 1 - \cos t}{t^3}.$$
(c) The longest interval on which the solution of the given initial value problem is defined is: $(-1, 1)$. The solution of the given initial value problem is:
$$y = \frac{1+x}{1-x}.$$

2. In what follows mass is measured in lb, volume in gal, and time in seconds. Let $M(t)$ be the mass of dissolved salt in the tank at time t.

(a) The time T when the tank just begins to overflow is 300 s. The initial value problem that governs the mass of salt in the tank from $t = 0$ to the instant $t = T$ is as follows:
$$\frac{dM}{dt} = 5 - \frac{3M}{100+t},$$
$$M(0) = 50.$$
(b) $M(t) = \dfrac{5}{4}(100+t) - \dfrac{75 \times 10^6}{(100+t)^3}.$

(c) For $t > T$, $M(t)$ satisfies the differential equation
$$\frac{dM}{dt} = 5 - \frac{4M}{400}$$
or
$$\frac{dM}{dt} = 5 - \frac{M}{100}.$$
For $t \geq T$, the mass of dissolved salt in the tank is given by
$$M(t) = 500 + (M(T) - 500)e^{\frac{-1}{100}(t-T)},$$
where $T = 300\,\mathrm{s}$, and $M(T)$ is computed from the formula in (b) as $498\frac{53}{64}$ lb.

3. (a) $y_p = t^2 \left(\frac{1}{2}t + 1\right) e^{-t}$; $y = c_1 e^{-t} + c_2 t e^{-t} + y_p$.

   (b) $y_p = -\frac{1}{5} e^t \cos 3t$; $y = c_1 e^t \cos 2t + c_2 e^t \sin 2t + y_p$.

4. (a) $y_p = x(Ax^2 + Bx + C)\cos 3x + x(Dx^2 + Ex + F)\sin 3x + G\cos 2x + H\sin 2x$.

   (b) $y_p = Axe^{-x}\cos x + Bxe^{-x}\sin x + Ce^x \cos x + De^x \sin x$.

5. (a) Since $y_1 = t$, $y_1' = 1$ and $y_1'' = 0$, we have $t^2 y_1'' + t y_1' - y_1 = 0$.

   (b) $y_2 = 1/t$.

   (c) Since $W(y_1, y_2) = -2/t \neq 0$ for each $t \in (0, \infty)$, $\{y_1, y_2\}$ constitutes a fundamental set of solutions of the given homogeneous equation.

   (d) $y = c_1 t + c_2/t - 4t^{1/2}/3$.