MA 214 002 Calculus IV (Spring 2016)

Review Problems for Exam 1

1. Find the solution of the following initial value problem:

$$t^2\frac{dy}{dt} - ty = t^3 + 2, \qquad y(-1) = 2.$$

Determine the largest interval on which the solution is defined.

2. Solve the following initial value problems:

(a) $\dfrac{dy}{dx} = 4x^3y - y, \quad y(1) = -3.$

(b) $(x^3 + \dfrac{y}{x})dx + (y^2 + \ln x)dy = 0, \quad y(1) = 0.$

Is equation (a) linear? Is it separable? Is it exact? Answer the same questions for equation (b).

3. It is known that the differential equation

$$x^2 + y + f(x)\frac{dy}{dx} = 0$$

becomes exact when multiplied by the integrating factor $\mu(x) = x$.

(a) Find the differential equation that the function f must satisfy.

(b) Find all possible functions $f(x)$.

4. A 30-year-old woman accepts an engineering position with a starting salary of \$50,000 per year. Her salary $S(t)$ increases exponentially, with $S(t) = 50\,e^{t/20}$ *thousand* dollars after t years. Meanwhile 10% of her salary is deposited in a retirement account, at an annual rate of return of 5%.

(a) Formulate the initial-value problem that governs $A(t)$, the amount in her retirement account after t years.

(b) Determine $A(40)$, the amount available for her retirement at age 70. You may leave your answer in terms of e or take $e \approx 2.718$.

5. A 100-m^3 tank is full of water that contains a pollutant Z at a concentration of 0.5 g/m^3. Cleaner water, with a pollutant concentration of 0.1 g/m^3, is pumped into the well-mixed tank at a rate of 4 m^3/hour. Water flows out of the tank through an overflow valve at the same rate as it is pumped in. Determine the amount and concentration of pollutant in the tank as a function of the time elapsed since cleaner water is pumped in.

6. A 2000-liter tank is initially *half* full of water containing 10 kg of salt in solution. Water containing salt at a concentration of 0.2 kg/L enters at a rate of 5 L/h. An open valve allows the well-mixed solution to leave at 2 L/h. Water evaporates from the tank at 1 L/h.

   (a) Find the time t_f at which the tank just begins to overflow, and determine the volume $V(t)$ of solution in the tank as a function of the time t, for $0 \le t < t_f$.

   (b) Determine the concentration of the solution in the tank at $t = t_f$.

7. Consider the initial-value problem

$$\frac{dy}{dt} + \sqrt{1 - y^2} = 0, \qquad y(0) = 1.$$

   (a) Show that $y_1(t) \equiv 1$ and $y_2(t) = \cos t$ both satisfy the given initial-value problem. Why does this fact not contradict Theorem 2.4.2 of Boyce and DiPrima's text on existence and uniqueness of solution for first-order equations?

   (b) Give the region of points (t_0, y_0) in the ty-plane for which the aforementioned theorem guarantees the existence of a unique solution $y(t)$ that satisfies the given differential equation and the initial condition $y(t_0) = y_0$.

8. Determine the equilibrium solutions of the differential equation

$$\frac{dy}{dt} = 7y - y^2 - 10,$$

   and classify each one as asymptotically stable, unstable, or semistable. Draw the phase line. For the initial condition $y(0) = y_0$, sketch several graphs of solutions in the t-y plane for various values of y_0 in $(-\infty, \infty)$. Explain why a solution curve of the given equation cannot meet any other solution curve in the t-y plane.

9. Find the general solution of each of the following differential equations:

   (a) $y'' + 5y' = 0$;

   (b) $y'' - 4y' + 4y = 0$;

   (c) $y'' - 2y' + 2y = 0$.

10. (a) Use Euler's formula to write $\exp((1 - 2i)t)$, where $i = \sqrt{-1}$ and t is a real parameter, in the form $a + ib$.

   (b) Solve the following initial value problem:

$$y'' + 2y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$