MA 214 002 Calculus IV (Spring 2016)

Answers to Review Problems for Exam 1

1. The solution is: $y = t^2 - \dfrac{1}{t}$. The largest interval on which the solution is defined is: $(-\infty, 0)$.

2. (a) The given differential equation is both linear and separable. The solution of the initial-value problem is: $y = -3\, e^{x^4 - x}$.

   (b) The differential equation is exact. The solution of the given initial-value problem is:
$$\frac{x^4}{4} + y \ln x + \frac{y^3}{3} = \frac{1}{4}.$$

3. (a) The differential equation that the function f must satisfy is:
$$\frac{df}{dx} + \frac{1}{x}\, f = 1.$$

   (b) All possible functions f are given by the formula
$$f(x) = \frac{x}{2} + \frac{C}{x},$$
   where C is an arbitrary constant.

4. (a) The initial-value problem that governs $A(t)$, the amount in thousands of dollars, is:
$$\frac{dA}{dt} - 0.05\, A = 5\, e^{t/20}, \qquad A(0) = 0.$$

   (b) $A(40) = 200 \times e^2 \approx 1478$ thousands.

5. The mass $M(t)$ of pollutant Z in the tank at elapsed time t since cleaner water is pumped into the tank is given by the formula
$$M(t) = 10 + 40e^{-t/25},$$
   where M is in grams and t in hours. The concentration $c(t)$ of pollutant Z is the tank at time t is given by
$$c(t) = 0.1 + 0.4e^{-t/25},$$
   where c is in $\mathrm{g/m^3}$.

6. (a) $t_f = 500$ hours; $V(t) = 1000 + 2t$, where V is in liters and t in hours.

   (b) At $t = t_f$, the concentration of the salt solution in the tank is 0.19 kg/L.

7. (a) Let $f(t, y) = -\sqrt{1 - y^2}$. Then $\partial f/\partial y = y(1 - y^2)^{-1/2}$. There does not exist any open rectangle $(\alpha, \beta) \times (\gamma, \delta)$ that contains $(0, 1)$ in the ty-plane in which both f and $\partial f/\partial y$ are continuous. Hence the sufficient condition in Theorem 2.4.2 that guarantees the uniqueness of solution is not satisfied. Thus the fact that both y_1 and y_2 are solutions of the given initial-value problem does not contradict Theorem 2.4.2.

   (b) The required region is the open unit disc $\{(t, y) : t^2 + y^2 < 1\}$.

8. The equilibrium solutions are: $y(t) = 2$, which is unstable, and $y(t) = 5$, which is asymptotically stable. The phase line and typical solution curves can be sketched with information given in the following table, where $f(y) = 7y - y^2 - 10$:

	$(-\infty, 2)$	$(2, 7/2)$	$(7/2, 5)$	$(5, \infty)$
$y' = f(y)$	$-$	$+$	$+$	$-$
$f'(y)$	$+$	$+$	$-$	$-$
$y'' = f'(y)f(y)$	$-$	$+$	$-$	$+$
Concavity of solutions $y(\cdot)$	D	U	D	U

Table 1: Concavity of solutions $y(\cdot)$; D = concave down, U = concave up.

By the theorem on the uniqueness of solution for initial-value problems of first-order differential equations, a solution curve of the given equation cannot meet any other solution curve in the t-y plane.

9. (a) $y = c_1 + c_2 e^{-5t}$.

   (b) $y = c_1 e^{2t} + c_2 t e^{2t}$.

   (c) $y = c_1 e^t \cos t + c_2 e^t \sin t$.

10. (a) $e^{(1-2i)t} = e^t \cos 2t - i e^t \sin 2t$.

    (b) $y = e^{-t} \cos \sqrt{3}t + \dfrac{1}{\sqrt{3}} e^{-t} \sin \sqrt{3}t$.