MA 113 - Calculus I
FINAL EXAM

Spring 2002
04/29/2002

Name: ____________________ Sec.: ____

SEC.	INSTRUCTORS	T.A.'S	LECTURES	RECITATIONS
001	A. Corso	B. Bennewitz	MWF 8:00-8:50, CB 204	TR 8:00-9:15, CB 341
002	A. Corso	B. Bennewitz	MWF 8:00-8:50, CB 204	TR 9:30-10:45, CB 345
004	M. Silhavy	H. Song	MWF 10:00-10:50, CB 214	TR 8:00-9:15, CB 349
005	M. Silhavy	C. Budovsky	MWF 10:00-10:50, CB 214	TR 2:00-3:15, CB 343
006	M. Silhavy	H. Song	MWF 10:00-10:50, CB 214	TR 3:30-4:45, CB 345
007	A. Martin	M. Neu	MWF 12:00-12:50, CB 208	TR 9:30-10:45, CB 347
008	A. Martin	Y. Jia	MWF 12:00-12:50, CB 208	TR 11:00-12:15, CB 347
009	A. Martin	Y. Jia	MWF 12:00-12:50, CB 208	TR 12:30-1:45, CB 349
010	M. Silhavy	C. Budovsky	MWF 2:00-2:50, CB 204	TR 12:30-1:45, CB 345
011	M. Silhavy	M. Slone	MWF 2:00-2:50, CB 204	TR 2:00-3:15, CB 345
012	M. Silhavy	M. Slone	MWF 2:00-2:50, CB 204	TR 3:30-4:45, CB 349

Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,
- clearly indicate your answer and the reasoning used to arrive at that answer (*unsupported answers may receive NO credit*).

QUESTION	SCORE	TOTAL
1.		16
2.		16
3.		6
4.		10
5.		8
6.		16
7.		12
8.		8
9.		8
TOTAL		100

1. Compute the following limits. Each limit is worth 4 points.

(a) $\lim_{x\to 2} \frac{x^3-1}{x-1} =$ $\lim_{x\to 2} (x^2+x+1) = 7$

recall $x^3-1 = (x-1)(x^2+x+1)$

(b) $\lim_{x\to 0} \frac{\cos^2 x - 1}{2x^2} =$ $\lim_{x\to 0} -\frac{1}{2}\frac{\sin^2 x}{x^2} = -\frac{1}{2}\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = -\frac{1}{2}$

$\sin^2 x + \cos^2 x = 1 \implies \cos^2 x - 1 = -\sin^2 x$

$\downarrow$

1

(c) $\lim_{x\to\infty} \cot\left(\frac{2}{x}+\frac{\pi}{4}\right) =$ $\cot(\pi/4) = 1$

$\cot x = \frac{\cos x}{\sin x}$ as $x \to \infty$ $\frac{2}{x}+\frac{\pi}{4} \longrightarrow \frac{\pi}{4}$

$\therefore \cot(\pi/4) = 1$

(d) $\lim_{x\to\infty} \frac{\sqrt{9x^4+8}}{3x^2+\sqrt{x}} =$ 1

$$\lim_{x\to\infty} \frac{\frac{\sqrt{9x^4+8}}{x^2}}{\frac{3x^2+\sqrt{x}}{x^2}} = \lim_{x\to\infty} \frac{\sqrt{9+\frac{8}{x^4}}}{3+\sqrt{1/x^3}} =$$

$$= \frac{\sqrt{9}}{3} = 1$$

pts: /16

2. Find the derivative of the following functions. Each derivative is worth 4 points. Do **not** simplify your answers.

(*a*) If $f(x) = (2x^8 + 7)(3x^2 + 5x)$ then $f'(x) =$ ____________

$$f'(x) = 16x^7(3x^2 + 5x) + (2x^8+7)\cdot(6x+5)$$

(*b*) If $f(x) = \sin(\sqrt[3]{x^2})$ then $f'(x) =$ $\frac{2}{3\sqrt[3]{x}} \cdot \cos(\sqrt[3]{x^2})$

$f(x) = \sin(x^{2/3})$ use chain rule

$f'(x) = \cos(x^{2/3}) \cdot \frac{2}{3} \cdot x^{-1/3}$

(*c*) If $f(x) = \cos^3(x^3) + (5x^2 - 3)^3$ then $f'(x) =$ ____________

$$f'(x) = 3\cos^2(x^3)\cdot(-\sin(x^3))\cdot 3x^2 + 3(5x^2-3)^2\cdot 10x$$

(*d*) If $f(x) = \dfrac{\sin(x^3 - 1)}{x^3 + 1}$ then $f'(x) =$ ____________

use quotient rule and chain rule

$$f'(x) = \frac{[\cos(x^3-1)\cdot 3x^2](x^3+1) - \sin(x^3-1)\cdot 3x^2}{(x^3+1)^2}$$

pts: /16

3. A particle is moving on a line such that its position after t hours is $s(t) = -t^2 + t + 2$ measured in miles.

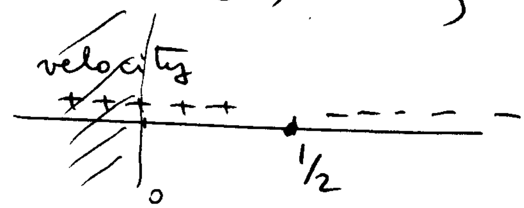
(*a*) Find the velocity of the particle.

$v(t) = -2t + 1$

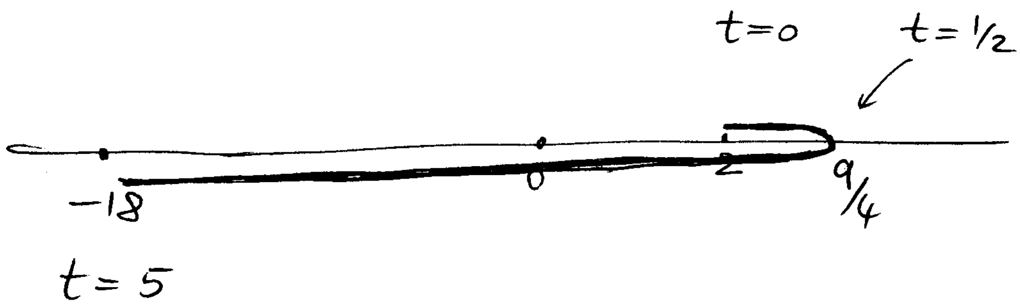
(*b*) When does the particle change its direction?

it changes direction when $v(t) = 0$,

i.e. at $t = 1/2$



(*c*) What is the largest distance of the particle to its origin within the first 5 hours.



motion of the particle

the position at $t=0$ is 2;
the position at $t=1/2$ is $9/4$.
The position at $t=5$ is -18.

pts: /6

Thus the largest distance to its origin is (in absolute value) $\underline{20}$

4. Consider the function $f(x) = \dfrac{x^3}{x^2-4}$.

(*a*) (3pts) Determine the intervals where $f(x)$ is increasing or decreasing. Find the values of f at the local minima and maxima of f.

$$f'(x) = \frac{3x^2(x^2-4) - x^3(2x)}{(x^2-4)^2} = \frac{x^4 - 12x^2}{(x^2-4)^2}$$

$+\ +\ \underset{-\sqrt{12}}{\bullet}\ -\ \underset{-2}{\otimes}\ -\ -\ -\ \underset{0}{\bullet}\ -\ -\ -\ \underset{2}{\otimes}\ -\ \underset{\sqrt{12}}{\bullet}\ +\ +$

$$f'(x) = 0 \iff x^4 - 12x^2 = 0 \quad x^2(x^2-12) = 0 \quad x = 0,\ x = \pm\sqrt{12}$$

(*b*) (3pts) Determine the intervals where $f(x)$ is concave up or down. Find the values of f at its inflection points.

$$f''(x) = \frac{(4x^3-24x)(x^2-4)^{\cancel{2}} - (x^4-12x^2)\cdot 2\cancel{(x^2-4)}\cdot 2x}{(x^2-4)^{\cancel{4}}} = \frac{8x^3+96x}{(x^2-4)^3} = \frac{8x(x^2+12)}{(x^2-4)^3}$$

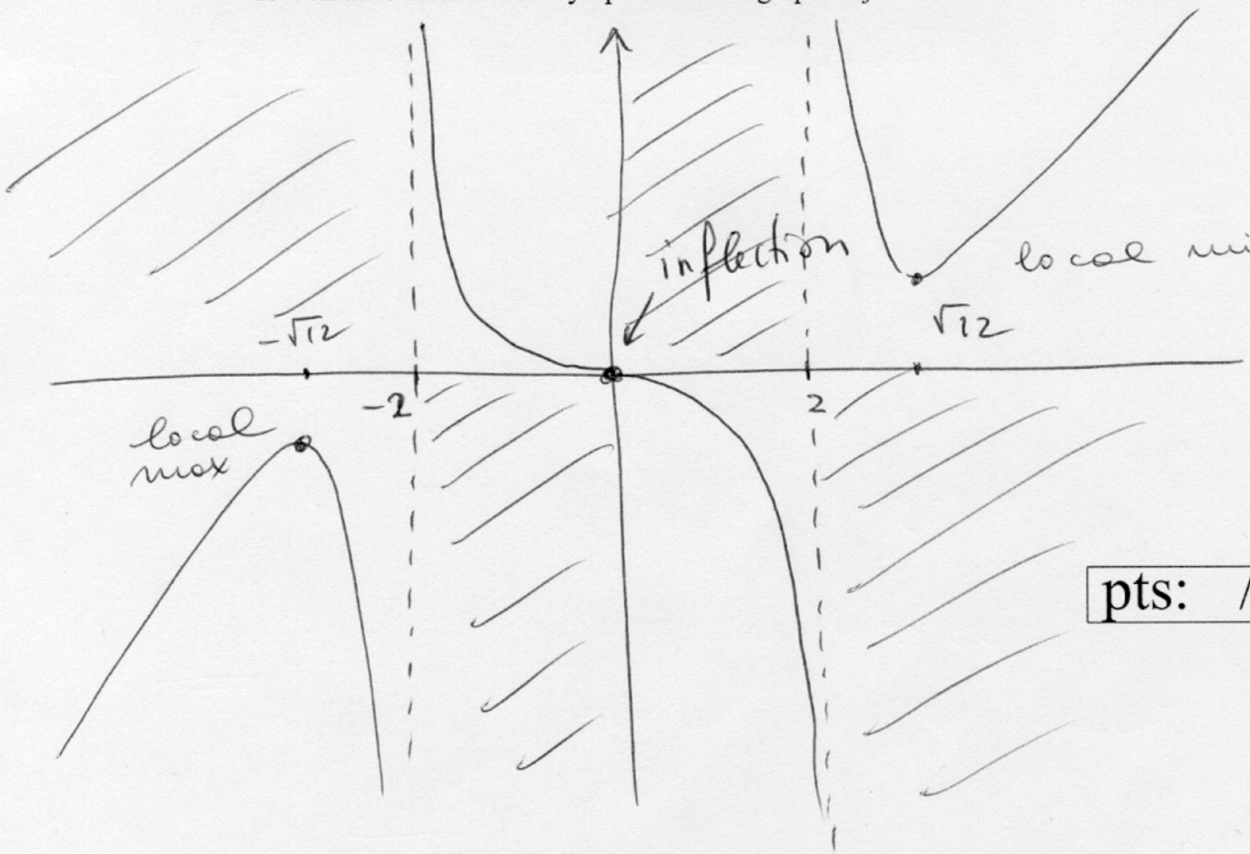
sign of f'': $-\ -\ \underset{-2}{\otimes}\ +\ \overset{0}{\bullet}\ -\ \underset{2}{\otimes}\ +\ +\ +$

(*c*) (2pts) Find the horizontal and vertical asymptotes of the graph of f.

vertical asymptotes $x = 2,\ x = -2$; $\displaystyle\lim_{x\to+\infty} \frac{x^3}{x^2-4} = +\infty$

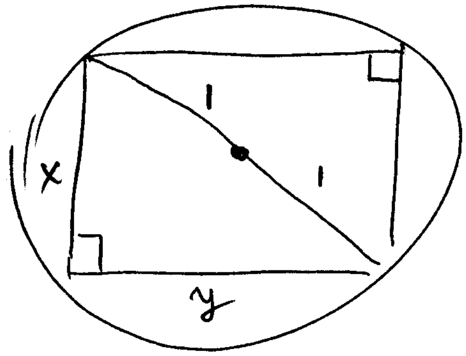
$\therefore$ no horizontal asymptote. $\displaystyle\lim_{x\to-\infty} \frac{x^3}{x^2-4} = -\infty$

(*d*) (2pts) Sketch the graph of f. **Make sure** to label the local extrema and the inflection points as well as to include the asymptotes of the graph of f.



pts: /10

5. Find the largest area of a rectangle that can be inscribed in a circle of radius 1.



Notice that the main diagonals of any such rectangle are diameters.

Now, if x and y are the dimensions of the rectangle we have that $x^2+y^2=2^2=4$

thus $y=\sqrt{4-x^2}$. We need to maximize the function

$$\text{Area} = x\cdot y = x\sqrt{4-x^2} \qquad 0 \le x \le 2$$

$$A' = 1\cdot\sqrt{4-x^2} + x\cdot\frac{1}{\cancel{2}}\frac{(-\cancel{2}x)}{\sqrt{4-x^2}} = \frac{4-x^2-x^2}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}}$$

$$A'=0 \iff 4-2x^2=0 \qquad x^2=2 \qquad x=\pm\sqrt{2}$$

x	$A(x)$
0	0
2	0
$\sqrt{2}$	2

value at the end points

max

pts: /8

Notice $x=\sqrt{2}=y$ $\therefore$ it is a square

6. Find the following indefinite integrals. Each problem is worth 4 points.

(*a*) $\int (\sqrt{x}+\sin(5x))\,dx =$ $\boxed{\tfrac{2}{3} x\sqrt{x} - \tfrac{1}{5}\cos(5x) + \text{Const}}$

$$\int \sqrt{x}\,dx + \int \sin(5x)dx = \frac{2}{3}x^{3/2} + \frac{1}{5}(-\cos(5x)) + \text{const}$$

$$x^{3/2} = x\sqrt{x}$$

(*b*) $\int \sqrt[3]{x}\cdot(x^7-1)\,dx =$ $\frac{3}{25}x^8\sqrt[3]{x} - \frac{3}{4}x\sqrt[3]{x} + \text{Const}$

$$\int (x^{7+1/3} - x^{1/3})dx = \int (x^{22/3} - x^{1/3})dx = \frac{3}{25}x^{\frac{25}{3}} - \frac{3}{4}x^{4/3} + \text{Const}$$

$$x^8 x^{1/3} \qquad x\cdot x^{1/3}$$

(*c*) $\int \frac{\sin x}{\cos^5 x}\,dx =$ $\frac{1}{4\cos^4 x} + \text{Const}$

use $u = \cos x$

$du = -\sin x\,dx$

$$= \int -\frac{du}{u^5} = -\int u^{-5}du = -\frac{1}{-4}u^{-4} + \text{const}$$

$$= \frac{1}{4u^4} + \text{Const}$$

(*d*) $\int (x^6+x^3)^7\cdot(2x^5+x^2) =$ $\frac{1}{24}(x^6+x^3)^8 + \text{Const}$

set $u = x^6+x^3$

$du = (6x^5+3x^2)\,dx$

$\therefore \frac{du}{3} = (2x^5+x^2)\,dx$

$$\therefore = \int u^7\frac{du}{3} = $$

$$= \frac{1}{3}\cdot\frac{1}{8}u^8 + \text{Const}$$

pts: /16

NOTE: there was a typo in problem (d) !!

7. Calculate the following definite integrals. Each problem is worth 4 points.

(*a*) $\int_0^9 (x^2 - \sqrt{x})\, dx =$ 225

$$= \frac{1}{3}x^3 - \frac{2}{3}x^{3/2}\Big]_0^9 = \frac{1}{3}9^3 - \frac{2}{3}9^{3/2} - 0 = 243 - 18$$

$$= 225$$

THERE WAS A TYPO

(*b*) $\int_0^1 x^3\sqrt{1+2x^4}\, dx =$ $\frac{3\sqrt{3}-1}{12}$

$u = 1+2x^4$

$du = +8x^3\,dx$

$+\frac{1}{8}du = x^3 dx$

$$= +\frac{1}{8}\int_1^{+3} \sqrt{u}\,du = \frac{1}{8}\int_1^3 u^{1/2}\,du = \frac{1}{8}\left[\frac{2}{3}u^{3/2}\right]_1^3$$

$$= \frac{1}{8}\cdot\frac{2}{3}\left[3\sqrt{3} - 1\right] = \frac{3\sqrt{3}-1}{12}$$

(*c*) $\int_0^1 (2x+1)^2\, dx =$ $13/3$

$u = 2x+1$

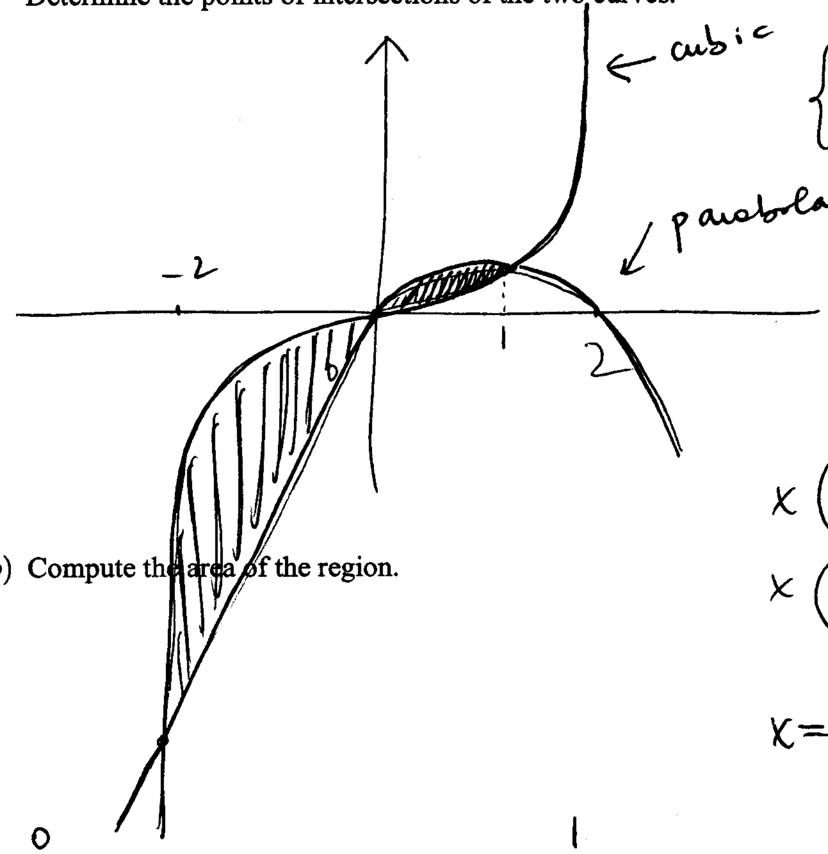
$du = 2\,dx$

$$= \int_1^3 \frac{1}{2}u^2\,du = \frac{1}{6}u^3\Big]_1^3 =$$

$$= \frac{1}{6}\left[27 - 1\right] = \frac{26}{6} = \frac{13}{3}$$

pts: /12

8. (*a*) Sketch the region that is bounded by the graphs of the functions $f(x) = 2x - x^2$ and $g(x) = x^3$. Determine the points of intersections of the two curves.



$$\begin{cases} y = x^3 \\ y = 2x - x^2 \end{cases}$$

$$x^3 = 2x - x^2$$

$$x^3 + x^2 - 2x = 0$$

$$x(x^2 + x - 2) = 0$$

$$x(x+2)(x-1) = 0$$

$$x = 0, 1, -2$$

(*b*) Compute the area of the region.

$$\text{Area} = \int_{-2}^{0} \left(x^3 - (2x - x^2)\right) dx + \int_{0}^{1} \left((2x - x^2) - x^3\right) dx$$

$$= \int_{-2}^{0} (x^3 - 2x + x^2) dx + \int_{0}^{1} (-x^3 - x^2 + 2x)\, dx =$$

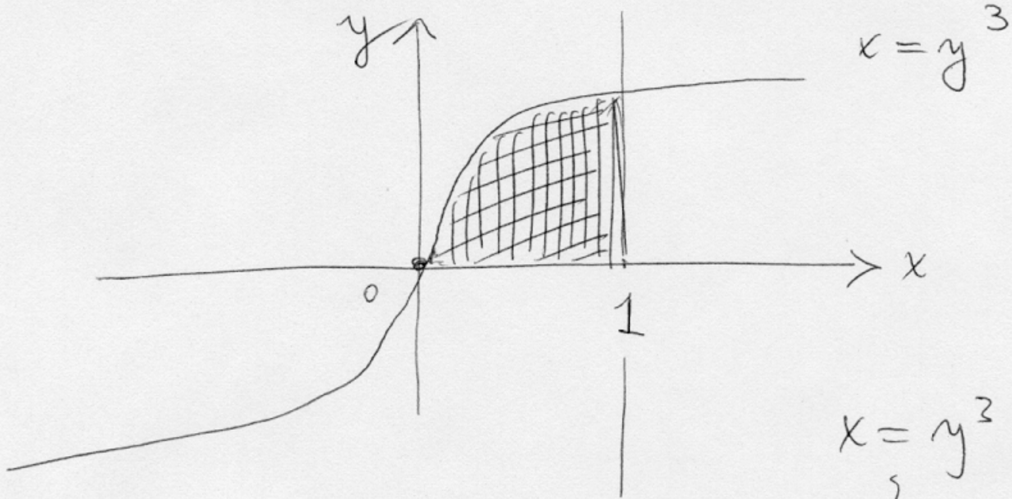
$$= \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2\right]_{-2}^{0} + \left[-\frac{1}{4}x^4 - \frac{1}{3}x^3 + x^2\right]_{0}^{1} =$$

$$= 0 - \left(\cancel{4} + \frac{1}{3}(-8) - \cancel{4}\right) + \left(-\frac{1}{4} - \frac{1}{3} + 1\right)$$

$$= \frac{8}{3} + \frac{-3-4+12}{12} = \frac{32+5}{12} = \frac{37}{12}$$

pts: /8

9. Find the volume of the solid that is obtained by rotating about the x-axis the region bounded by the curve $x - y^3 = 0$ and the line $x = 1$.



$$x = y^3$$
$$\downarrow$$
$$y = f(x) = \sqrt[3]{x}$$

$$\text{Volume} = \int_0^1 \pi \left[f(x)\right]^2 dx$$

$$= \int_0^1 \pi \left[\sqrt[3]{x}\right]^2 dx = \int_0^1 \pi \; x^{2/3} \, dx$$

$$= \pi \cdot \frac{3}{5} x^{5/3} \Big]_0^1 = \pi \cdot \frac{3}{5} \cdot 1^{5/3} - 0$$

$$= \frac{3}{5}\pi$$

pts: /8