MA 113 - Calculus I
THIRD MIDTERM

Spring 2002
04/09/2002

Name: ______________________ Sec.: ____

SEC.	INSTRUCTORS	T.A.'S	LECTURES	RECITATIONS
001	A. Corso	B. Bennewitz	MWF 8:00-8:50, CB 204	TR 8:00-9:15, CB 341
002	A. Corso	B. Bennewitz	MWF 8:00-8:50, CB 204	TR 9:30-10:45, CB 345
004	M. Silhavy	H. Song	MWF 10:00-10:50, CB 214	TR 8:00-9:15, CB 349
005	M. Silhavy	C. Budovsky	MWF 10:00-10:50, CB 214	TR 2:00-3:15, CB 343
006	M. Silhavy	H. Song	MWF 10:00-10:50, CB 214	TR 3:30-4:45, CB 345
007	A. Martin	M. Neu	MWF 12:00-12:50, CB 208	TR 9:30-10:45, CB 347
008	A. Martin	Y. Jia	MWF 12:00-12:50, CB 208	TR 11:00-12:15, CB 347
009	A. Martin	Y. Jia	MWF 12:00-12:50, CB 208	TR 12:30-1:45, CB 349
010	M. Silhavy	C. Budovsky	MWF 2:00-2:50, CB 204	TR 12:30-1:45, CB 345
011	M. Silhavy	M. Slone	MWF 2:00-2:50, CB 204	TR 2:00-3:15, CB 345
012	M. Silhavy	M. Slone	MWF 2:00-2:50, CB 204	TR 3:30-4:45, CB 349

Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,
- clearly indicate your answer and the reasoning used to arrive at that answer (*unsupported answers may receive NO credit*).

QUESTION	SCORE	TOTAL
1.		10
2.		10
3.		15
4.		8
5.		10
6.		15
7.		10
8.		10
9.		12
TOTAL		100

1. Find all the critical values and the absolute maximum and absolute minimum values for

$$f(x) = 3x^4 - 16x^3 + 18x^2$$

on the closed interval $-1 \leq x \leq 4$.

	x	$f(x)$
end points	-1	37
	4	32
critical values	0	0
	1	5
	3	-27

$f'(x) = 12x^3 - 48x^2 + 36x$

$f'(x) = 0 \iff x^3 - 4x^2 + 3x = 0$

$x(x^2 - 4x + 3) = 0$

$x(x-3)(x-1) = 0$

$\therefore\ x = 0,\ x = 1,\ x = 3$

$\therefore$ $f(x)$ has an absolute max of 37 at $x = -1$

$f(x)$ has an absolute min of -27 at $x = 3$

pts: /10

2. (*a*) Does the Mean Value Theorem apply to the function $f(x) = \dfrac{x+1}{x-1}$ on the interval $2 \leq x \leq 3$? Why? If so, find all possible values of c for which the Mean Value Theorem holds on the given interval.

the function $f(x)$ is continuous and differentiable on $[2,3]$. The MVT says that there exists $c \in (2,3)$ s/t

$f'(c) = \dfrac{f(3) - f(2)}{3-2} = \dfrac{4/2 - 3/1}{1} = -1$

Note that $f'(x) = \dfrac{1\cdot(x-1) - 1\cdot(x+1)}{(x-1)^2} = -2/(x-1)^2$

$\therefore\ -\dfrac{2}{(c-1)^2} \overset{\text{WANT}}{=} -1$

$\therefore\ (c-1)^2 = 2$

$\therefore\ c = 1 \pm \sqrt{2}$

But $1-\sqrt{2}$ is outside the interval,

so $\boxed{c = 1+\sqrt{2}}$

(*b*) Same as (*a*), but on the new interval $0.5 \leq x \leq 1.5$.

the mean value theorem does not apply as the function is not continuous at $x = 1 \in [0.5, 1.5]$

pts: /10

3. Consider the function: $f(x) = x^4(x^2-3).$ $= x^6 - 3x^4$

Each question is worth 5 points.

(*a*) Determine the intervals where the graph of $f(x)$ is increasing or decreasing.
Find the values of $f(x)$ at the local maxima and minima of $f(x)$.

$$f'(x) = 6x^5 - 12x^3 = 6x^3(x^2-2) = 0$$

$\Longleftrightarrow$ $x = 0$ or $x = \pm\sqrt{2}$

decr.		incr.		decr.		increas
$- - -$		$+ + +$		$- - -$		$+ + + +$
$\searrow$	$-\sqrt{2}$	$\nearrow$	0	$\searrow$	$\sqrt{2}$	$\nearrow$

local min at $x = \pm\sqrt{2}$

$f(\pm\sqrt{2}) = -4$

local max at $x = 0$

$f(0) = 0$

(*b*) Determine the intervals where the graph of $f(x)$ is concave up or down.
Find the values of $f(x)$ at the inflection points of $f(x)$.

$$f''(x) = 30x^4 - 36x^2 = 6x^2\left[5x^2 - 6\right] = 0$$

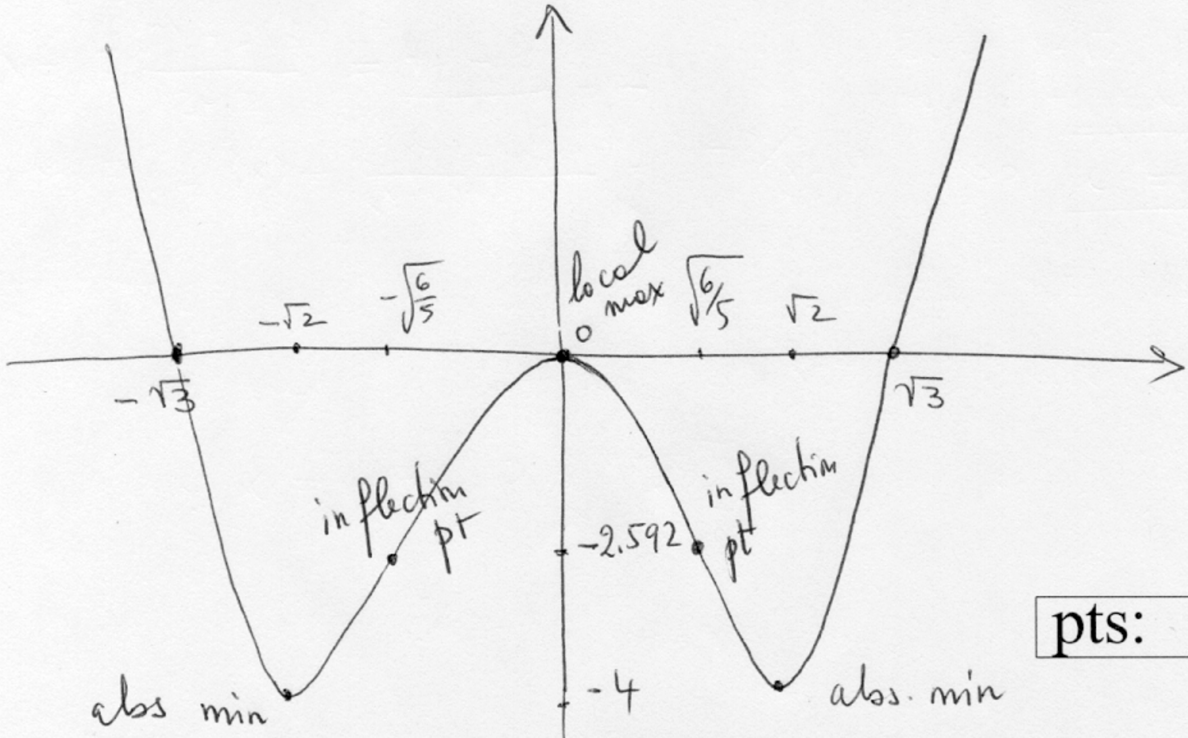
$\Longleftrightarrow$ $x = 0,\ x = \pm\sqrt{\frac{6}{5}}$

$+ + +$	$-\sqrt{\frac{6}{5}}$	$- - -$		$- - - -$	$\sqrt{\frac{6}{5}}$	$+ + + + +$
Conc up		conc. down				Conc up

$x = \pm\sqrt{\frac{6}{5}}$ inflection pts

$f\left(\pm\sqrt{\frac{6}{5}}\right) = -2.592$

(*c*) Sketch the graph of $f(x)$.
Make sure to label the local maxima, the local minima and the inflection points of $f(x)$.



pts: /15

4. Without using a calculator, show that the equation

$$x^{101} + x^{51} + x - 1 = 0$$

has exactly one real root.

EXISTENCE

Observe that if we consider $f(x) = x^{101} + x^{51} + x - 1$ defined on $[0,1]$, then by the Intermediate Value Theorem $f(x)$ has a root in $(0,1)$. In fact $f(x)$ is continuous and $f(0) = -1$, while $f(1) = 2$.

UNIQUENESS

Suppose that $f(x)$ has 2 roots "a, b". I.e. $f(a) = 0 = f(b)$. Since f is continuous and differentiable by ROLLE'S THEOREM there exist $c \in (a,b)$ when $f'(c) = 0$. BUT $f'(x) = 101x^{100} + 51x^{50} + 1$ is never zero

$\therefore$ there is only one real root !!!

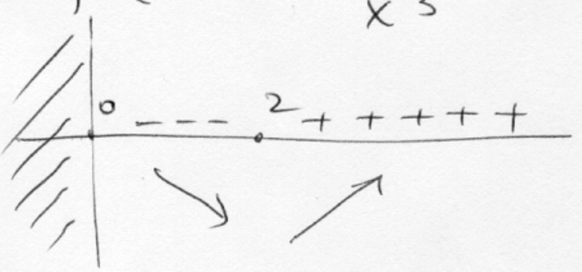
pts: /8

5. Show that if $x > 0$ then $x + \frac{4}{x^2} \geq 3$.

Let $f(x) = x + \frac{4}{x^2}$ defined on the half line $x > 0$. Notice that $f'(x) = 1 - \frac{8}{x^3}$

also, f has only one critical value for $x > 0$.

$$0 = f'(x) = \frac{x^3 - 8}{x^3} \iff x^3 - 8 = 0 \iff x = 2$$



sign of $f'(x)$

$\therefore$ f has an absolute min at $x = 2$

$$\therefore f(x) = x + \frac{4}{x^2} \geq f(2) = 2 + \frac{4}{2^2} = 3$$

pts: /10

6. Each question is worth 5 points.

(*a*) $\lim_{x\to\infty} \frac{\sqrt{x}+3}{3-2x} =$ $\boxed{0}$

$$= \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}} + \frac{3}{x}}{\frac{3}{x} - 2} = \frac{0}{-2} = 0$$

(*b*) $\lim_{x\to\infty} \frac{2\sqrt{1+9x^2}}{9-16x} =$ $-3/8$

$$= \lim_{x\to\infty} \frac{\frac{2\sqrt{1+9x^2}}{x}}{\frac{9-16x}{x}} = \lim_{x\to\infty} \frac{2\sqrt{\frac{1}{x^2}+9}}{\frac{9}{x}-16} = \frac{2\sqrt{9}}{-16} = -\frac{6}{16} = -\frac{3}{8}$$

(*c*) Find the vertical and horizontal asymptotes of the curve

$$f(x) = \frac{3x^2+4}{2-x^2}.$$

Compute $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ for all the values of 'a' such that the line $x = a$ is a vertical asymptote of the given function $f(x)$.

the equation of the horizontal asymptot is $y = -3$ as $\lim_{x\to\infty} \frac{3x^2+4}{2-x^2} = -3$

The function has 2 vertical asymptotes at $\boxed{x = \sqrt{2},\ x = -\sqrt{2}}$.

$\lim_{x\to\sqrt{2}^+} f(x) = -\infty$

$\lim_{x\to\sqrt{2}^-} f(x) = +\infty$

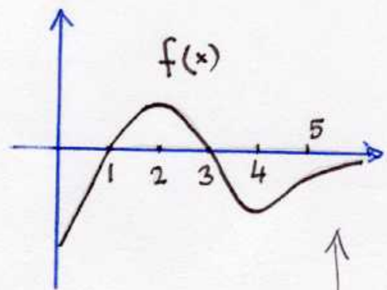
$\lim_{x\to-\sqrt{2}^+} f(x) = +\infty$

$\lim_{x\to-\sqrt{2}^-} f(x) = -\infty$

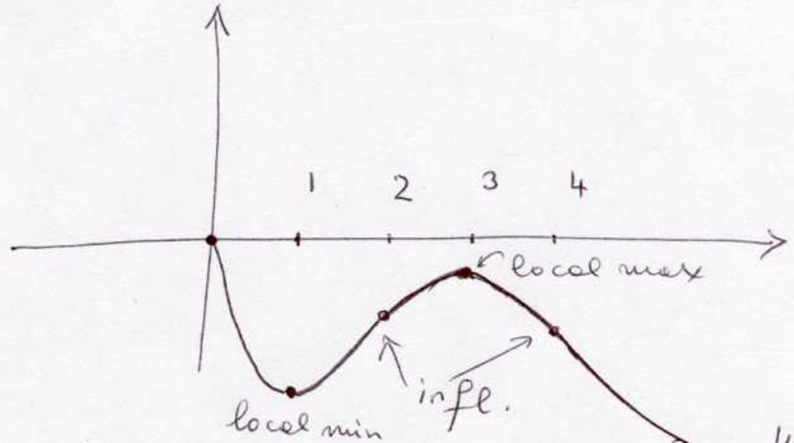
pts: /15

7. Each problem is worth 5 points.

(a) The graph of a function $f(x)$ is shown. Which graph is an antiderivative of $f(x)$ and why?



let's graph the antiderivative $F(x)$ such that $F(0)=0$



(b) Find the most general antiderivative of: $f(x) = x^3 + \sqrt{x} - 2\cos(2x). = x^3 + x^{1/2} - 2\cos(2x)$

$$f(x) = \frac{1}{4}x^4 + \frac{2}{3}x^{3/2} - \sin(2x) + \underline{\underline{\text{Const}}}$$

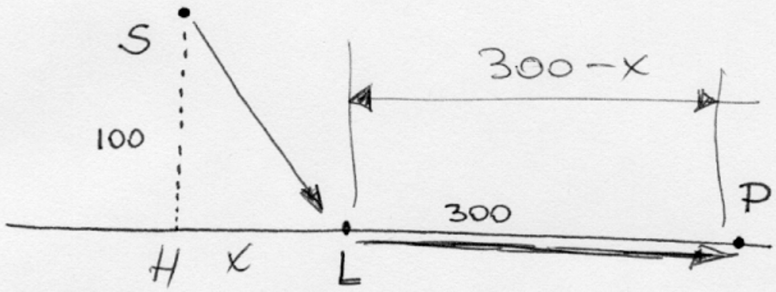
don't forget it

$$\boxed{= \frac{1}{4}x^4 + \frac{2}{3}x\sqrt{x} - \sin(2x) + \text{Const}}$$

pts: /10

8. A swimmer S is in the ocean 100 meters from a straight shoreline. A person P in distress is located on the shoreline 300 meters from the point on the shoreline closest to the swimmer.

$$\text{time} = \frac{\text{space}}{\text{velocity}}$$



If the swimmer can swim 3 meters per second and run 5 meters per second, what path should the swimmer follow in order to reach the person in distress as quickly as possible?

Let x be the distance between H and L (L = landing point): $0 \le x \le 300$

We need to minimize the time to go from S to P (via L).

$$T(x) = \text{time} = \underbrace{\frac{\sqrt{x^2+100^2}}{3}}_{\text{time to go from } S \text{ to } L} + \underbrace{\frac{300-x}{5}}_{\text{time to go from } L \text{ to } P}$$

$$T'(x) = \frac{1}{3} \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+10000}} - \frac{1}{5} = 0$$

$$\Longleftrightarrow \quad \frac{x}{3\sqrt{x^2+10000}} = 1/5$$

$$\Longleftrightarrow \quad 5x = 3\sqrt{x^2+10000}$$

$$\Longleftrightarrow \quad 25x^2 = 9x^2 + 90000 \quad \therefore 16x^2 = 90000$$

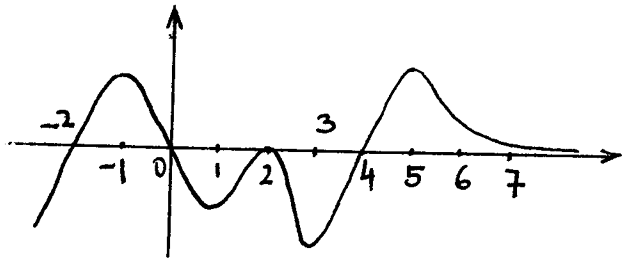
$$x = \pm\sqrt{\frac{90000}{16}}$$

But $x = +75$

	x	$T(x)$
end points	0	93.3 sec
	300	105.4 sec
critical value	$\boxed{75}$	$\boxed{86.6 \text{ sec}}$

pts: /10

9. The graph of the derivative $f'(x)$ of a function $f(x)$ is shown:



Each question is worth 3 points.

(a) On what intervals is $f(x)$ increasing or decreasing?

decreasing   increas.   decreasing   increasing

$- -$ $+ + +$ $- - - -$ $- - -$ $+ + + + +$ sign of f'

-2 0 2 4

$\searrow$ $\nearrow$ $\searrow$ $\searrow$ $\nearrow$

(b) At what values of x does $f(x)$ attains a local maximum or minimum?

$f(x)$ has local max at $x = 0$

$f(x)$ has local min at $x = -2, 4$

(c) On what intervals is $f(x)$ concave up or down?

sign of f'' $+ +$ $- - - -$ $+ +$ $- -$ $+ +$ $- - - - - -$

-1 1 2 3 5

(d) State the x-coordinates of the inflection points.

$f(x)$ has inflection points at the points with x- coordinates:

$x = -1, 1, 2, 3, 5$

pts: /12