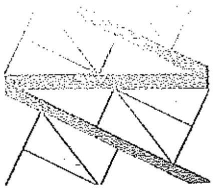
4.6

The Shape of Reality?

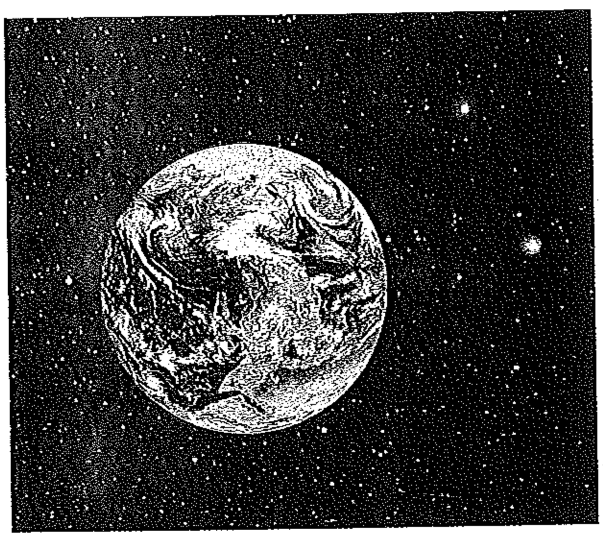
How Straight Lines Can Bend in Non-Euclidean Geometries



If there is anything that can bind the heavenly mind of man to this dreary exile of our earthly home and can reconcile us with our fate so that one can enjoy living—then it is verily the enjoyment of the mathematical sciences and astronomy.

—Johannes Kepler

Mathematics can help us understand the cosmic, the unapproachable, and the mysterious. Nothing is more cosmic and mysterious than the entire universe. For thousands of years, people have pondered the fundamental question: What is the shape of our universe?



In any attempt to understand the world around us, it is only natural to wonder about the geometry of our physical existence. Of course, our universe is incredibly vast, and our experience is limited by time and space. Thus our question is by no means easy. Does space bend or curve? What does it even mean for space to bend or curve? Since we do not see space bending or curving around us, our initial sense is that space is flat. However, we exist on such a microscopic scale compared to that of the entire universe, perhaps we don't sense the reality of the "big picture." So let's apply some of our techniques of analysis and try to discover the shape of space.

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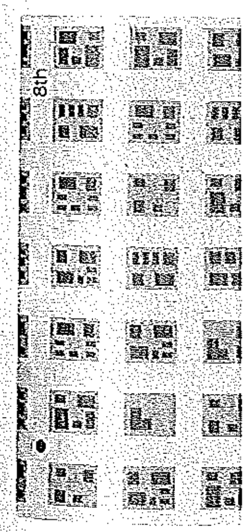


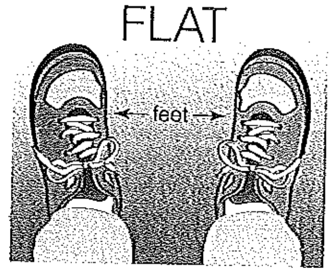
Manhattan

A First Sketch of the "Big Picture"

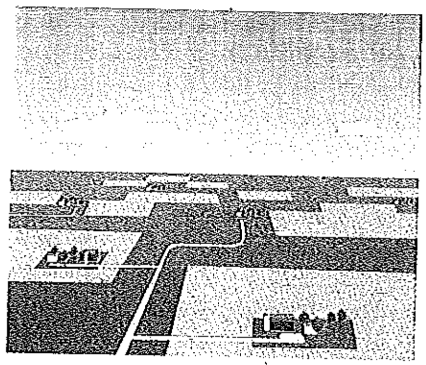
How do we start to understand the geometry of something so large that it is beyond our understanding? Start by looking at something smaller.

In dealing with life's complex issues, start with the simple and build from there.

First, we look at the ground under and just around our feet. What do we see? Flat.



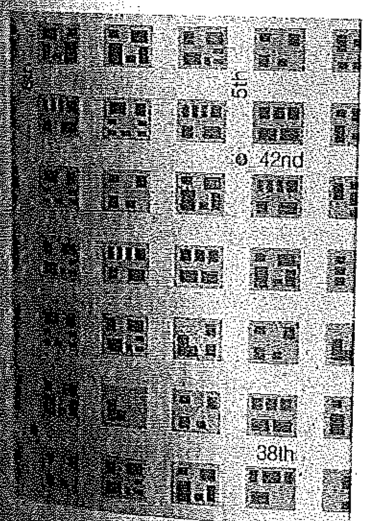
Thus, it seems reasonable to guess that our world around us is flat like a plane. This guess is not completely crazy, especially to those who live in Kansas. The world around us does tend to look pretty flat. This observation led people throughout history to study the flat plane and its rich geometry. However, it turns out that our Earth is shaped like a ball. This nontrivial fact illustrates two important points. First, there is no pressing need for the Flat Earth Society; and second, what we observe locally may not accurately depict what is occurring on a larger scale.



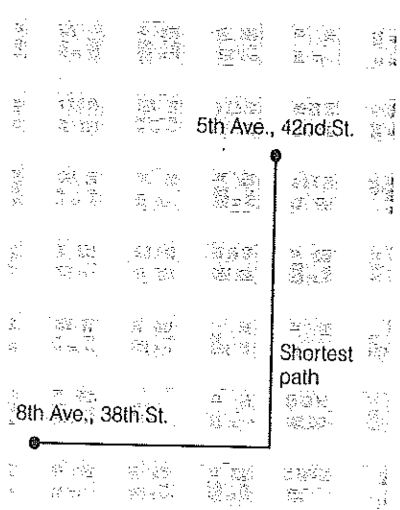
Before taking on the whole universe, perhaps we should consider the geometry of the next simplest realm: the sphere.

A Next Sketch: The Geometry of a Sphere

What is the shortest distance between two points? In a flat, unobstructed world, that shortest distance is always a straight line. But in New York City the shortest distance from 5th Ave. and 42nd St. to 8th Ave. and 38th St. is not a straight line. What path does the crow take? If the crow drove a taxi, he would find the shortest path, but that path would follow the grid of streets. So the shortest paths between two points—the "straight lines"—depend on the shape of the space where we live. We live on Earth. So what are the "straight lines" on Earth? Let's travel around and see.



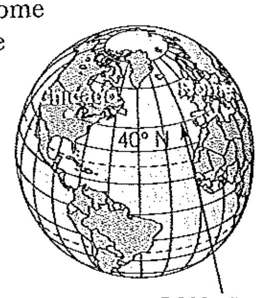
Manhattan



Since the Earth is round, we do not live on a plane; yet many travelers live in a plane a good deal of the time. Pilots have a great attachment to fuel and hate to run out of it at 35,000 feet in the air. Thus, airplanes go from place to place along the shortest routes possible. Pilots, like crows, know the Earth is round and choose their routes accordingly. Let's see what those routes are. The best method for bringing this point home would be for you to now take a nonstop plane trip from Chicago, Illinois to Rome, Italy. We'll wait patiently, but you had better send us a postcard.

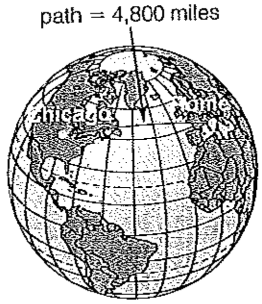
Traversing Your Travels

Chicago and Rome are both at the latitude of nearly 40° north. You might think that the shortest route from Chicago to Rome would be to stick to the 40° latitude line the whole way. Let's measure how far that route would be. We will do this by measuring distances on a globe and using the scale to tell the mileage. If we take out a tape measure and place it along the 40° latitude line, we see that the distance is 5,300 miles. Is there a shorter route? If we're flying the plane, we had better find out.



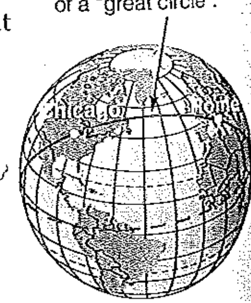
path shown = 5,300 miles

A good, though messy, way to find the shortest route uses a greased globe and a rubber band. We take the globe and grease it until it is so slippery that nothing, including the rubber band, will stick to it. We next put two pins in the globe, one at Chicago and one in Rome, and then stretch a rubber band over the two pins. We first hold the rubber band down so it sits on the latitude line. Then we let go. Did it stay on the 40° latitude line? We don't think so. In fact, it slid up to a shorter route. Instead of flying along the latitude, the rubber band found a genuinely shorter route. Notice that the route heads north and goes over Labrador and Dublin, Ireland before heading back south on its way to Rome.

rubber band's path = 4,800 miles


Is this route really shorter? Let's measure. We placed our measuring tape along our new route and measured about 4,800 miles. This new route saves about 500 miles!

Let's take a closer look at this rubber-band route. If we extend the route, we get a *great circle* that is as big as possible going around the whole globe—that is, a circle whose center is at the center of the globe. It is as long as the equator.

The arc path is part of a "great circle".


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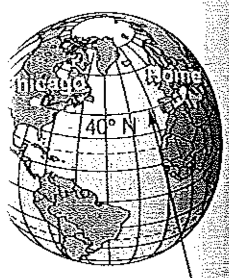

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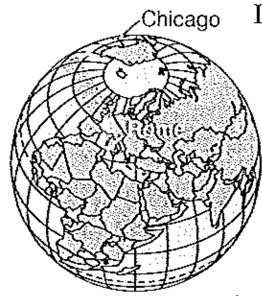
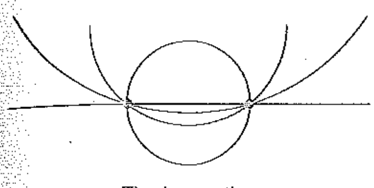
The arc path is part of a "great circle".

Indeed, the shortest path between any two points on the globe is always a great circle that contains them. The segments of great circles are the shortest distance between two points on the globe. Why?

Why Great Circles Are the Way to Go



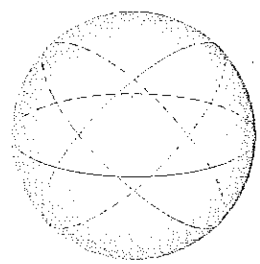
Let's think about why the great circle segments are the shortest paths. If the Earth were hollow, the shortest path from Chicago to Rome really would be a straight line inside the Earth. So for our purposes, let's imagine a straight-line tunnel connecting Chicago and Rome burrowing right through the Earth. The shortest route on the surface of the Earth would deviate as little as possible from that straight, underground Chicago-Rome tunnel.



The larger the circle—the "flatter" the segment.

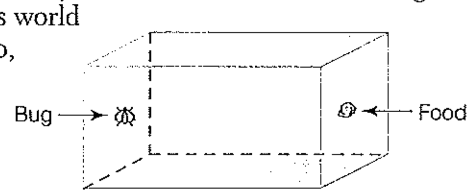
Let's notice something about circles and lines. If we take two points and make a circle that contains them, then bigger circles are flatter and therefore stay closer to the straight line between the two points. So, among the paths that stay on circles, taking the biggest circle on the globe containing Chicago and Rome—that is, the *great circle*—will stay closest to the straight line. The latitude circle, being smaller, bends out more from the straight line and is therefore longer. "Straight lines," that is, the shortest paths on Earth, are great circle segments.

Latitude is longer.



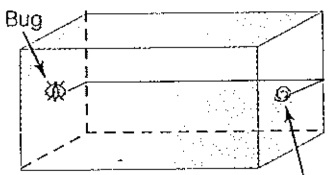
Distances in a Different World

We live on Earth, which is essentially a ball, but how about a bug on the wall? If our bug doesn't fly, its world is in the shape of the walls. So, when it sees its dining destination on some other wall, it has some serious calculations to perform. What is the shortest distance from here to dinner? Take a guess.



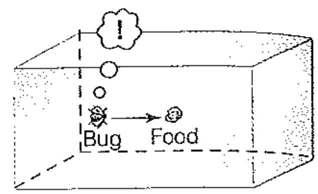
shortest path?

A good guess would be the path shown in the figure on the left.



an initial guess for the shortest path

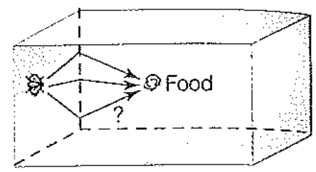
Is this guess the shortest path? Suppose dinner is on the same wall. Then the situation is easy. The bug simply walks in a straight line.



shortest path = straight line

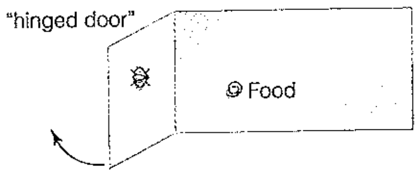
The bug is off to a great start. How about if dinner is on an adjacent wall? It is pretty clear that the bug needs to go straight to the boundary edge and then straight on the next wall to its dinner. The question is, "Where on that edge should it cross?" Describe a method for locating the best crossing place.

Think about some simple cases.



where to cross the edge?

Notice that, if the walls were at some angle other than 90°, the distances from the bug to the crossing point and the crossing point to dinner do not change. So let's consider a different question. Suppose the bug is on an open door, and its dinner is on the wall.

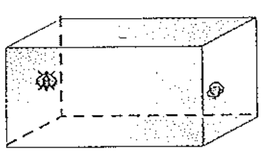


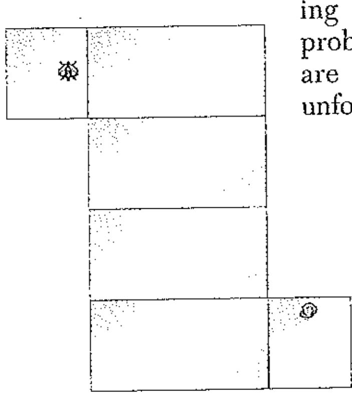
shortest path!

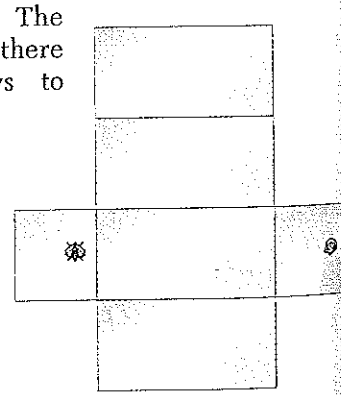
Imagine that, as the bug is considering its shortest path, someone comes along and closes the door. Suddenly the question becomes much easier. Now the bug and its dinner are on the same wall, and the bug simply proceeds in a straight line. Now the bug is on a roll.

Let's now return to the scenario where the bug's dinner is on the opposite wall. How will it figure out the shortest route? Having experienced the closing of the door, surely our bug cannot resist the idea of unfolding the walls. The problem is that there are many ways to unfold the walls.

Can you think of other ways of unfolding the walls, keeping the food and the bug in the same relative locations?







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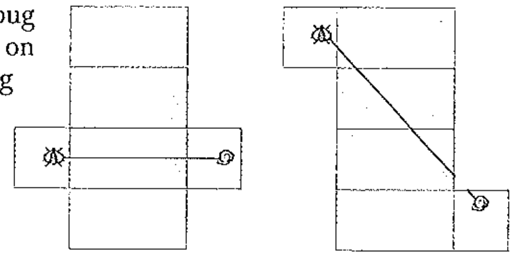
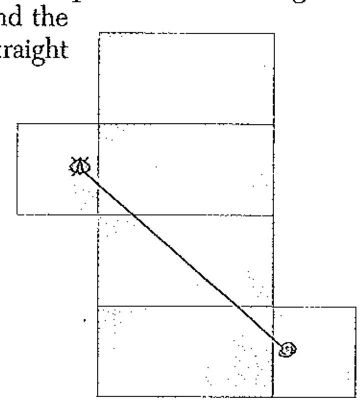
Which one should the bug choose? The straight lines on some unfolded walls from bug to dinner are different lengths from others.

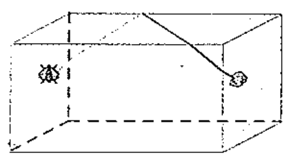
What to do?



Some straight paths are shorter than others.

One method would be to unfold the room in all possible ways and measure the straight-line distances. We've seen several different room-unfolding possibilities. Notice that different unfolding scenarios result in different placements of the food and the bug. Visualize the reassembly of the flattened rooms and verify that the relative positions of the bug and the food are always the same. Once we find the shortest flattened path, we can draw the straight line on that unfolded version and then reassemble the room. In this case, the shortest path takes the bug over five walls—a dramatic departure from our first guess.





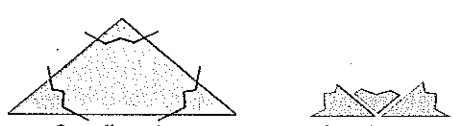
shortest path

Sometimes, when we are faced with a problem and we don't know what to do, we should just consider everything.

Now we have a better sense of shortest paths and straight lines in various worlds, including our own earthly sphere. Putting these straight lines together allows us to explore some basic geometry that captures the essence of the graceful curvature of the sphere. Let's put three straight lines together to make a triangle.

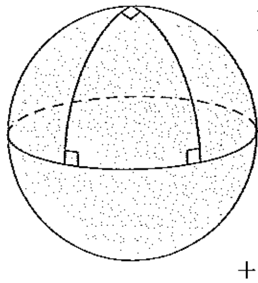
Triangles on the Sphere

Draw any triangle in the plane. Add up the three angles. The result is 180°.



Cut off angles. Sum is 180°.

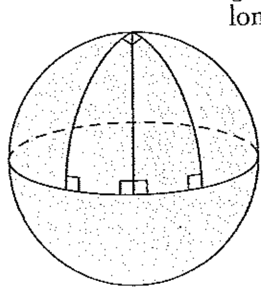
But we've seen that, in different realms, we have different ideas of straightness. Let's now explore the angles of a triangle made out of straight lines on a sphere. We begin by drawing a large triangle on a sphere. For exam-



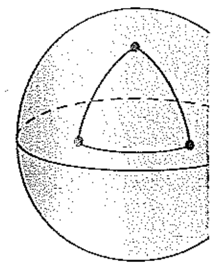
ple, let's put one vertex on the north pole, one vertex on the equator at 0° longitude, and the third vertex on the equator at 90° longitude. The edges of this triangle consist of two longitudinal segments from the north pole to the equator and a segment that goes ¼ of the way around the equator. What are the angles at each vertex? Each one is 90°. So, what is the sum of the three angles of that triangle on the sphere? 90° + 90° + 90° = 270°. Yikes.

This result is slightly disconcerting. The sum of the angles of this triangle on the sphere is not 180°, but 270° (90° too much). Is it possible that all triangles on the sphere have angles that sum to 270°? Let's see.

Take the big triangle above and break it into two by drawing in the longitudinal segment from the north pole down to the equator at 45°.



Now each of the half-sized triangles has angles of 90°, 90°, and 45°. That sum is 225°; 45° more than the 180° we would have in a triangle on the flat plane. This result is stranger still, since not only do the angles fail to add up to the comfortable 180° we know and love, but now we see that on a sphere, different triangles have different sums of angles. As always, we must look for patterns.



Curvature of the spr
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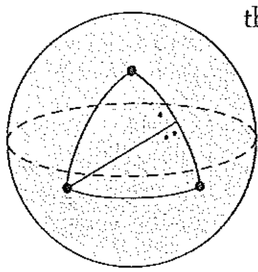
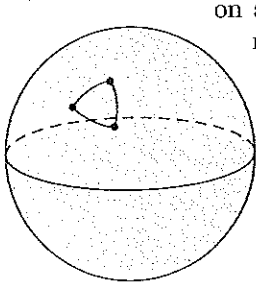
Can we find any regularity among our measurements? The big triangle had 270°, 90° degrees too much. When we divided it in half, each half had 225°, 45° degrees more than 180°. Did you notice that the total surplus of angle for the two smaller triangles stayed at 90°? In other words, when we took the big triangle and measured the surplus angle bigger than 180°, we got 90°. Then, when we divided the big triangle into two smaller triangles, each of the halves had a surplus of 45°, or 90° altogether. Suppose we start with any triangle on a sphere and cut it in half by bisecting one of the angles.



•° + ••° = 180°

What is the relationship between the angles of the original triangle and the angles of each of the two subtriangles? Well, the new angles created add up to 180° since they are on a straight line. So, the total excess of the two triangles must be equal to the excess for the original big triangle.



Sum is just a smidge greater than 180°.

Notice what happens if we take a small triangle on the sphere. What is the surplus of its angles? Not very much. A small part of a sphere is basically flat, so the angles of a triangle there will have almost exactly the same angles as the angles of a triangle on a flat plane. So, it seems that larger triangles have greater excess in the sum of their angles than small triangles do. Furthermore, if a large triangle is divided into smaller triangles by adding edges, since all the

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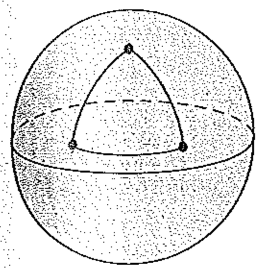


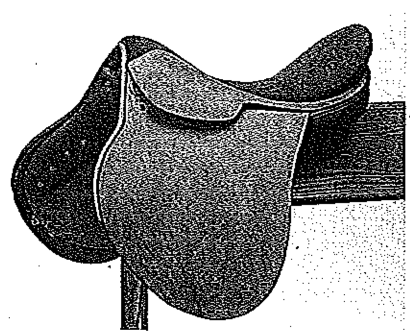
Curvature of the sphere causes "straight lines" to bow out a bit.

added angles created are along straight lines or divide existing angles, the total excess of all the subtriangles making up a bigger triangle must be the same as the excess of the big triangle. What corresponds to the excess? It turns out that the excess increases as the area of the triangle increases. Thus, we see that the sum of the angles of a triangle on a sphere will always exceed 180° but that small triangles will just barely exceed 180° and large triangles will exceed 180° by a more substantial amount.

Extra Degrees Through Curvature

The fact that the sum of the angles of any triangle on a sphere exceeds 180° is due to the curvature of the sphere. And, since every triangle has a sum of angles exceeding 180°, we will say the sphere has *positive curvature.* Notice that the curvature on a sphere can be determined by measurements taken on the sphere itself. It is not necessary to see the sphere from outside. For example, suppose we were bugs whose whole universe was a sphere. Perhaps light stayed right along the sphere so that we could see things. We would not see a horizon, because the light would bend around the sphere providing us with ever more distant vistas. Nevertheless, we could determine that our world has positive curvature by drawing a triangle and measuring the sum of the angles. Even though the individual lines would appear completely straight, the sum of the angles would be more than 180°, clinching the positive-curvature claim.

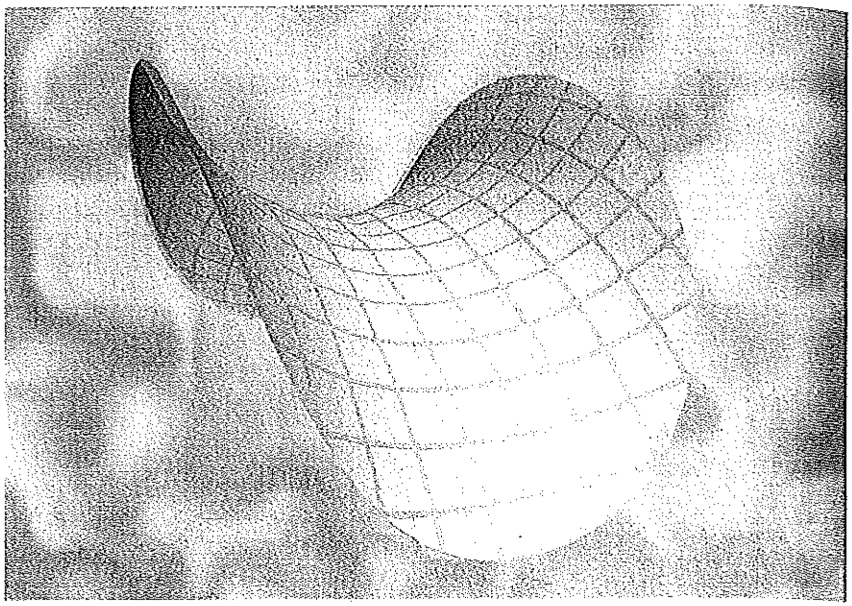
We now have one space, the plane, where all triangles have angles that add up to 180°. That constant sum is our benchmark, so we will say the plane has *zero curvature*—it is flat. We saw another space, the sphere, with positive curvature where all triangles have angles that add up to more than 180°. Surely we cannot resist asking the question, "Is there a space with *negative curvature*—that is, where the sum of the angles of a triangle is less than 180°?"



Geometry on a Saddle

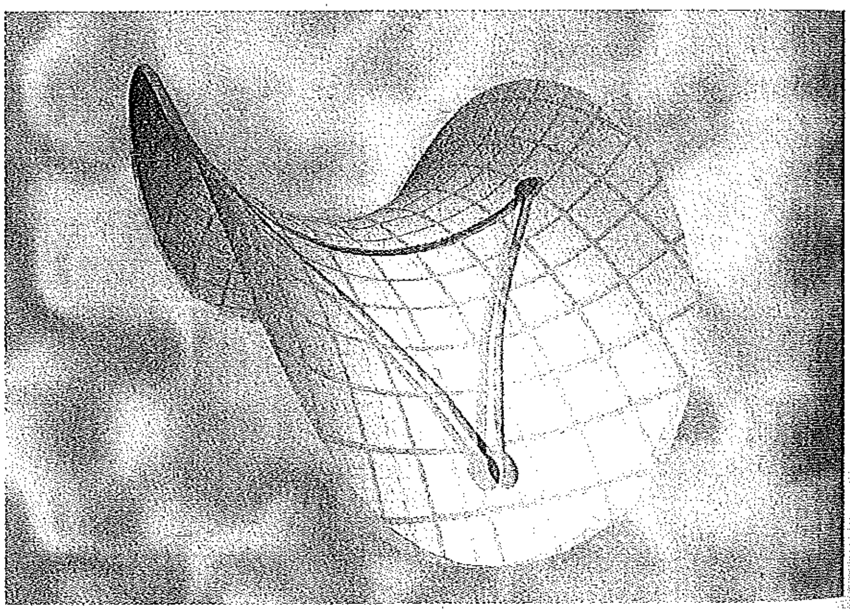
Horseback riding provides us not only with a sore bottom but also with an interesting geometrical opportunity. The surface of a saddle has an appealing shape and provides a surface ripe for experiments using rubber

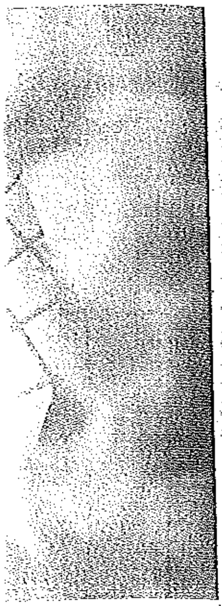
3D picture of a saddle
(Use your 3D glasses from your kit.)



bands and butter. Suppose we place three pins as shown, one near the front of the saddle and two near the stirrups. (This is a poor time to actually sit on the saddle.) We now grease the saddle with butter and put a rubber band around every pair of pins. The rubber bands will slide to the shortest distances between pins. So, we will have a rubber band triangle on the surface of the saddle. We now wish to measure the angles. If you don't happen to have a saddle handy, look at the following picture, estimate the measure of the angles, and compare the sum of the angles to

triangle on the surface of a saddle (Angles sum to less than 180°.)





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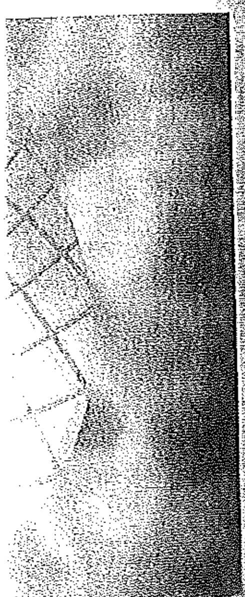
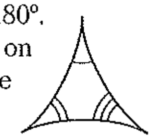


180°. The sum of the angles of this triangle is less than 180°. Of course, the rubber-band method will not always work on a saddle since the line between some pairs of points, like the center front to the center back of the saddle, would leave the surface and float in the air. However, rubber bands are good tools for finding the shortest distances between some pairs of points on the saddle. Whether you use rubber bands or another method for finding the shortest distances between points, triangles on the saddle will have sums of angles less than 180°, because the sides of the triangles curve inward and thus cause those angles to shrink. So this space has negative curvature and is an example of the exotic world known as *hyperbolic geometry*.



Sum of the angles is less than 180°.

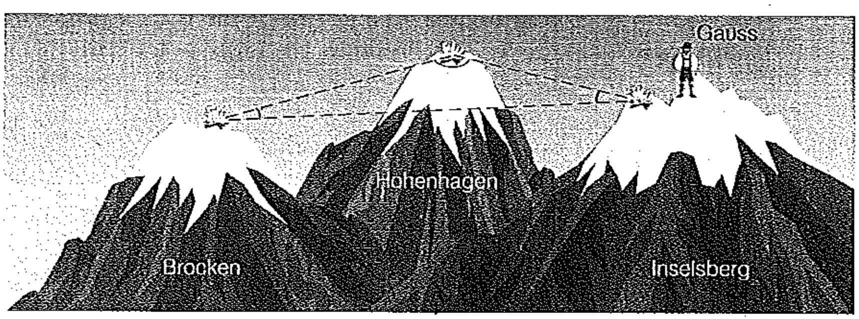
We have seen three different types of geometries: plane geometry, which we say has zero curvature (all triangles have angle sums of exactly 180°); spherical geometry, which we say has positive curvature (angle sums vary depending on the size of the triangle, but always exceed 180°); and hyperbolic geometry, which we say has negative curvature (angle sums vary depending on the size of the triangle, but always are less than 180°). It certainly appears as though hyperbolic geometry is exotic and foreign to our real-world existence, which brings us back to our original question: What is the shape of our universe?

The Shape of our Universe

We have just caught glimpses of three types of geometries: planar, spherical, and hyperbolic. Which is our universe? How could we tell which geometry accurately models our universe? Think of an experiment that we could perform to answer this. (*Hint:* What property distinguishes the three?)

Let's measure the angles of triangles. Suppose we make a big triangle and measure its angles. If the sum of those angles equals 180°, then we would conjecture that our universe has zero curvature. If the sum of those angles exceeds 180°, then we'd guess that our universe has positive curvature. If the sum of those angles is less than 180°, then we'd guess that our universe is curved negatively. Would anyone actually attempt this experiment? Yes!

The great mathematician Carl Friedrich Gauss tried this experiment in the early 1800s. He formed a triangle using three mountain peaks near Göttingen: Brocken, Hohenhagen, *und* Inselsberg. He had fires lit and



used mirrors to reflect the beams of light to form a triangle having side lengths roughly 43, 53, and 123 miles. He carefully measured the angles of the triangle and added them up. His sum was within 1/180 of a degree of 180°. That small difference could easily have been caused by errors in measurement. This evidence certainly leads us to think that our universe is neither positively nor negatively curved and that the universe is flat. What is the problem with this conclusion? Think about this question in view of our spherical geometry observations.

Recall that, in spherical geometry, if we have a small triangle, then the triangle is nearly flat, and thus the sum of its angles is nearly 180°. Thus, although Gauss's triangle was big, compared to the entire universe it wasn't even a speck. Thus, on such a microscopic scale, it is not surprising to see that the evidence points to a geometry having zero curvature. We would need an enormous triangle to detect the existence of any actual curvature. Is this experiment even practical? And if it were, would anyone actually attempt it?

The answer to the first question is possibly and to the second is yes. Today scientists believe that the universe exhibits two important properties. The first is that it is *homogeneous*, which basically means that any two large sections of space will look the same—of course here "large" needs to be LARGE. The second is that the universe is *isotropic*, which means that, as we look around, things look about the same in every direction. It turns out that we can find geometrical objects that are homogeneous and isotropic that are either planar, spherical, or hyperbolic. This fact leads to a question of great interest to scientists today: Does the universe have zero, positive, or negative curvature?

A large group of scientists now believe that the universe is negatively curved—that is, that the geometry of the universe is actually the exotic hyperbolic geometry suggested by the saddle. In fact, a conference was held in October 1997 at Case Western University that brought together 20 cosmologists and 5 mathematicians to discuss the possible shape of the universe and how to measure its curvature. NASA is scheduled in 2000 to send *MAP*—the Microwave Anisotropy Probe—into space. This probe will measure microwave background radiation, which is a residue of the "big bang." By studying slight variations in the measurements—which are actually temperature measurements—scientists are hoping to discover the exact geometry and curvature of the universe. It will take about two years to map out the heavens with the probe and another four years for scientists to analyze and collect the data. European scientists plan to send up the Planck Probe in about six years. This probe should be able to make even more careful measurements of the variations in microwave radiation. These modern experiments capture the spirit of Gauss's attempts to measure the curvature of the universe.

So, what is the shape of our universe? Although many experts believe it may be hyperbolic and negatively curved; no one knows for certain. However, 21st-century science and technology together with mathematics may enable us one day to measure the curvature of our vast space and understand its subtle and beautiful geometry.

A LOOK BACK

Space can have various shapes. We can distinguish how space bends by examining the shortest paths—straight lines, although they may not necessarily be straight. Three different kinds of geometry are planar, spherical, and hyperbolic. The flat plane, round sphere, and saddle are good models for planar, spherical, and hyperbolic geometries, respectively. On a very small scale all look nearly the same, and thus we have not yet been able to determine the shape of our universe by taking measurements of our local environment. However, some scientists now believe that the universe may be hyperbolic, and experiments are being devised to give evidence about the geometry of space.

The distinguishing feature of the three different geometries is their curvature. If a space has zero curvature (the sum of the angles of any triangle is exactly 180°), then the space is flat. If a space has positive curvature (the sum of the angles of any triangle exceeds 180°), then the space is spherical. Finally, if the space has negative curvature (the sum of the angles of any triangle is less than 180°), then the space is hyperbolic.

When we wish to consider big issues it is often valuable to start with simple and familiar models or examples and build from there. By identifying both similarities and differences in our various examples, we can often discover the underlying structure that determines the general case.

Start with the simple and build from there.

When you don't know what to do, consider everything.

Look for patterns.

MINDSCAPES Invitations to Further Thought*

I. SOLIDIFYING IDEAS

1–3. ***Travel agent.*** Get a globe and trace the shortest paths between the following pairs of cities. Match each pair in the left list with the location in the right list that is on the shortest path between them.

Austin, Texas—Tehran, Iran	Reykjavik, Iceland
Williamstown, Massachusetts—Beijing, China	Denali, Alaska
Austin, Texas—Beijing, China	near the north pole

*In the Mindscapes section, exercises marked (H) have hints for solutions at the back of the book. Exercises marked (S) have solutions.