Chapter Three

The Beginnings of Written Mathematics: Egypt

The Urban Revolution and Its African Origins

In the previous chapter we began our examination of early evidence of mathematical activity with an artifact found in the middle of Africa. For the next stage of our journey we remain on the same continent but move north to Egypt. Egypt is generally recognized as the homeland of one of the four early civilizations that grew up along the great river valleys of Africa and Asia over five thousand years ago, the other three being in Mesopotamia, India, and China. Egyptian civilization did not emerge out of the blue as a full-blown civilization without any African roots. This is supported by evidence of large concentrations of agricultural implements carbon-dated to around 13,000 BC, found during the UNESCO-led operations to salvage the ancient monuments of Nubia.

Although there are no tangible traces of the origins of these Neolithic communities, recent archaeological discoveries indicate that they may have belonged to groups from the once-fertile Sahara region who were forced to migrate, initially to the areas south and east, as the desert spread. So, just as Egypt was a "gift of the Nile" (in the words of Herodotus), the culture and people of Egypt were at least initially a "gift" of the heartlands of Africa, the inhabitants of which were referred to at times as "Ethiopians." This is borne out by the historian Diodorus Siculus, who wrote around 50 BC that the Egyptians "are colonists sent out by the Ethiopians. . . . And the large part of the customs of the Egyptians . . . are Ethiopian, the colonists still preserving their ancient manners" (Davidson 1987, p. 7).[1]

It is important that the African roots of the Egyptian civilization be emphasized so as to counter the still deeply entrenched view that the ancient Egyptians were racially, linguistically, and even geographically separated from Africa.[2] The work during the last fifty years, well summarized by Bernal (1987) and Davidson (1987), lays bare the flimsy scholarship and ideological bias of those who persist in regarding ancient Egypt as a separate entity, plucked out of Africa and replanted in the middle of the Mediterranean Sea.

What were the origins of the urban revolution that transformed Egypt into one of the great ancient civilizations? It is not possible to give a definitive answer. All we can do is surmise that the gradual development of effective methods of flood control, irrigation, and marsh drainage contributed to a significant increase in agricultural yield. But each of these innovations required organization. An irrigation system calls for digging canals and constructing reservoirs and dams. Marsh drainage and flood control require substantial cooperation among what may have been quite scattered settlements. Would it be too fanciful to conjecture that, before the emergence of the highly centralized government of pharaonic Egypt, a form of *ujamaa* (self-help communities)[3] may have come into existence as an institutional backup for these agricultural innovations? This may eventually have led to the establishment of administrative centers that grew into cities.[4]

Between 3500 and 3000 BC the separate agricultural communities along the banks of the Nile were gradually united, first to form two kingdoms—Upper and Lower Egypt—which were then brought together, in about 3100 BC, as a single unit by a legendary figure called Menes, who came from Nubia (part of present-day Sudan). Menes was believed to have founded a long line of pharaohs, thirty-two dynasties in all, who ruled over a stable but relatively isolated society for the next three thousand years. With the discovery of the Narmer Palette (dating back to the thirty-first century BC), some archaelogists have raised the possibility that Pharaoh Narmer predates Menes, which would then cast doubts on the traditional accounts. However, there are others who believe that Narmer and Menes are in fact the same person.

It is worth remembering that up to 1350 BC the territory of Egypt covered not only the Nile Valley but also parts of modern Israel and Syria. Control over such a wide expanse of land required an efficient and extensive administrative system. Censuses had to be taken, taxes collected, and large armies maintained. Agricultural requirements included not only drainage, irrigation, and flood control but also the parceling out of scarce arable land among the peasantry and the construction of silos for storing grain and other produce. Herodotus, the Greek historian who lived in the fifth century BC, wrote that

> Sesostris [Pharaoh Ramses II, c. 1300 BC] divided the land into lots and gave a square piece of equal size, from the produce of which he exacted an annual tax. [If] any man's holding was damaged by the encroachment of the river. . . . The King . . . would send inspectors to measure the extent of the loss, in order that he might pay in future a fair proportion of the tax at which his property had been assessed. Perhaps this was the way in which geometry was invented, and passed afterwards into Greece. (Herodotus 1984, p. 169)

He also tells of the obliteration of the boundaries of these divisions by the overflowing Nile, regularly requiring the services of surveyors known as *harpedonaptai* (literally "rope stretchers"). Their skills must have impressed the Greeks, for Democritus (c. 410 BC) wrote that "no one surpasses me in the construction of lines with proofs, not even the so-called rope-stretchers among the Egyptians." One can only suppose that "lines with proofs" in this context refers to constructing lines with the help of a ruler and a compass.

There were other pursuits requiring practical arithmetic and mensuration. As the Egyptian civilization matured, there evolved financial and commercial practices demanding numerical facility. The construction of calendars and the creation of a standard system of weights and measures were also products of an evolving numerate culture serviced by a growing class of scribes and clerks. And the high point of this practical culture is well exemplified in the construction of ancient Egypt's longest-lasting and best-known legacy—the pyramids.

Sources of Egyptian Mathematics

Time has been less kind to Egyptian mathematical sources recorded on papyri than to the hard clay tablets from Mesopotamia. The exceptional nature of the climate and the topography along the Nile made the Egyptian civilization one of the more agreeable and peaceful of the ancient world. In this it contrasted sharply with its Mesopotamian neighbors, who not only had a harsher natural environment to contend with but were often at the mercy of invaders from surrounding lands. Yet the very dryness of most of Mesopotamia, as well as the unavailability of any natural writing material, resulted in the creation of a writing medium that has stood the test of time far better than the Egyptian papyrus. However, it must be remembered that papyrus is quite a bit more durable than the palm leaves, bark, or bamboo used as writing materials by the ancient Chinese and Indians. It is interesting in this context to note that, owing to climatic conditions, almost all the papyri that survive are from Egypt and, even among these papyri, the ones that are best preserved belong to certain favored texts. It would therefore follow that basing one's impressions of ancient Egypt on these records could result in a skewed image of the society of that time.

There are two major sources and a few minor ones on early Egyptian mathematics. Most minor sources relate to the mathematics of a later period, the Hellenistic (332 BC to 30 BC) or Roman (30 BC to AD 395) periods of Egyptian history. The most important major source is the Ahmes (or Ahmose) Papyrus, named after the scribe who copied it around 1650 BC from an older document. It is also known as the Rhind Mathematical Papyrus, after the British collector who acquired it in 1858 and subsequently donated it to the British Museum. (Since we know in this instance who penned the document, it would be more proper to name it after the writer than the collector.) The second major source is the Moscow Papyrus, written in about 1850 BC; it was brought to Russia in the middle of the last century, finding its way to the Museum of Fine Arts in Moscow. Between them, the Ahmes and Moscow papyri contain a collection of 112 problems with solutions. At the time of the receipt of the Ahmes Papyrus by the British Museum in 1864, it was highly brittle with sections missing. A fortunate discovery of the missing fragments in the possession of the New York Historical Association in 1922 helped to restore it to its original form, although the two parts still remain with their separate owners.[5]

Other sources include the Egyptian Mathematical Leather Roll, from the same period as the Ahmes Papyrus, which is a table text consisting of twenty-six decompositions into unit fractions; the Berlin Papyrus, which contains two problems involving what we would describe today as simultaneous equations, one of second degree; the Reisner Papyri containing administrative texts from around 1900 BC, consisting of accounts of building construction and carpentry, including a list of workers arranged in groups needed for these activities; the Lahun mathematical fragments, formerly known as the Kahun Papyrus, also from around 1800 BC, containing six scattered mathematical fragments, all of which have now been deciphered;[6] and the Cairo Wooden Boards from the Middle Kingdom period. From a later period, there are the two ostraca texts (i.e., texts written on tiles/

potteries) from the New Kingdom and demotic texts from the Greek and Roman periods. The latter consists of one large papyrus, the Cairo Papyrus, plus six smaller texts plus several ostraca.

There is a third group of Egyptian mathematical texts that come from the last few centuries of the first millennium BC and the first half of the first millennium AD, all of which are written in Greek. A small subsection of texts in this group containing six ostraca, one papyrus roll, and three papyrus fragments are in some way related to Euclid's *Elements*. However, the majority in this group show little or no sign of having been influenced by Greek mathematics. Friberg (2005, p. vii) describes the manuscripts from this group as "non-Euclidean" mathematical texts.[7] And they constitute important evidence, as we shall see later, for tracing possible links between Egyptian, Babylonian, and Greek mathematics.

Ahmes tells us that his material is derived from an earlier document belonging to the Middle Kingdom (2025–1773 BC). There is even the possibility that this knowledge may ultimately have been derived from Imhotep (c. 2650 BC), the legendary architect and physician to Pharaoh Zoser of the Third Dynasty. The opening sentence claims that the Papyrus contains "rules for enquiring into nature, and for knowing all that exists, [every] mystery, . . . every secret." While an examination of the Ahmes Papyrus does not bear this out, it remains, with the tables and eighty-seven problems and their solutions, the most comprehensive source of early Egyptian mathematics, and it was more likely than not a teacher's manual. The Moscow Papyrus was composed (or copied) by a less competent scribe, who remains unknown. It shows little order in the arrangement of topics covered, which are not very different from those in the Ahmes Papyrus. It contains twenty-five problems, among them two notable results of Egyptian mathematics: the formula for the volume of a truncated square pyramid (or frustum), and a remarkable solution to the problem of finding what some interpreters consider to be the curved surface area of a hemisphere. Before looking in detail at the mathematics in these and other sources, we begin the next section with a discussion of the Egyptian system of numeration.

Three types of source materials on Egyptian mathematics can be distinguished: table texts, problem texts, and administrative texts. These texts were the product of a group of scribes, with their clearly defined hierarchy. An interesting glimpse into professional rivalry is shown in the Papyrus Anatasi from the New Kingdom in which one scribe taunts another:[8]

> You come here and [try] to impress me with your official status as "the scribe and commander of a work gang." Your arrogance and boastful behavior will be shown up by (how you tackle) the following problem: "A ramp, 730 cubits [long] and 55 cubits wide, must be built, with 120 compartments filled with reeds and beams.[9] It should be at a height of 60 cubits at its peak, 30 cubits in the middle, a slope of 15 cubits with a base of 5 cubits. The quantity of bricks required can be obtained from the troop commander." The scribes are all assembled but no one knows how to solve the problem. They put their faith in you and say: "You are a clever scribe, my friend! Solve [the problem] quickly for your name is famous. . . . Let it not be said: 'There is something he does not know.' Give us the quantity of bricks required. Behold, its measurements are before you; each of its compartments is 30 cubits [long] and 7 cubits [wide]."

It would seem that the problem set deals with four situations that require different calculations: (1) calculating the number of bricks needed to build a ramp; (2) calculating the number of persons needed to move an obelisk; (3) calculating the number of persons needed to erect a colossal statue; and (4) calculating the rations of a group of soldiers of a given size. It is not known whether the "arrogant" scribe solved the problem. However, for a modern reader, the data provided are insufficient to solve the problem, and hence a variety of interpretations have been suggested.[10]

It is clear in this instance that the task set for the scribe was a problem in practical mathematics. A number of other problems had little connection with real life. The teacher scribes were simply showing their student scribes how to apply certain procedures correctly. A scribe was either an instructor or an accountant. If he was an instructor, he was expected to teach "advanced" calculations to his students. If he was an accountant, he had to work out labor requirements, food rations, land allocation, grain distributions, and similar matters for his employers, who were either government officials or wealthy private individuals. It is usually an accountant scribe who is depicted on wall frescoes walking a few paces behind his master.

Number Recording among the Egyptians

From the beginning of the third millennium BC, there are records of names of persons and places as well as those of commodities and their quantities. An example of this is a mace head containing a list of tributes received by the pharaoh Narmer: 120,000 men, 400,000 oxen, and 1,422,000 goats.[11]

To record such large numbers would require a system of numerals that allowed counting to continue almost indefinitely by the introduction of a new symbol wherever necessary.

There is an impression, fostered (no doubt inadvertently) by many textbooks on the history of mathematics, that only one scheme of numeration was used in ancient Egypt: the hieroglyphic. This impression is quite consistent with a view of Egyptian civilization as stable and unchanging, with mathematics primitive yet sufficient to serve the economic and technological needs of the time. The truth is very different from this view. It is possible to distinguish three different notational systems—hieroglyphic (pictorial), hieratic (symbolic), and demotic (from the Greek word meaning "popular")—the first two of which made their appearance quite early in Egyptian history. The hieratic notation was employed in both the Ahmes and the Moscow papyri. It evolved into a script written with ink and a reed pen or other implements on papyrus, ostracon (tile/pottery), leather, or wood, changing from what earlier resembled the hieroglyphic script to a more cursive and variable style suiting the handwriting of the individual scribe. The demotic variant was a popular adaptation of the hieratic notation and became important during the Greek and Roman periods of Egyptian history.

The hieroglyphic system of writing was a pictorial script in which each character represented an object, some easily recognizable. Special symbols were used to represent each power of 10 from 1 to 10^7. Thus a unit was commonly written as a single vertical stroke, though when rendered in detail it resembled a short piece of rope. The symbol for 10 was in the shape of a horseshoe. One hundred was a coil of rope. The pictograph for 1,000 resembled a lotus flower, though the plant sign formed the initial *khaa*, the beginning of the Egyptian word "to measure." Ten thousand was shaped like a crooked finger which probably had some obscure phonetic or allegorical connotation. The stylized tadpole for 100,000 may have been a general symbol of large numbers. One million was shown by a figure with arms upraised, representing the god Heh supporting the sky. On rare occasions, 10 million was represented by the rising sun and possibly associated with Ra, the sun-god, one of the more powerful of the Egyptian deities.[12]

Thus the earliest Egyptian number system was based on the following symbols:

1 10 10^2 10^3 10^4 10^5 10^6 10^7

Any reasonably large number can be written using the above symbols additively, for example:

$$12{,}013 = 3 + 1(10) + 2(10^2) + 1(10^4) =$$ 

No difficulties arose from not having a zero or placeholder in this number system. It is of little consequence in what order the hieroglyphs appeared, though the practice was generally to arrange them from right to left in descending order of magnitude, as in the example above. While there were exceptions regarding the orientation of the number symbols in the case of the hieroglyphic numbers, the hieratic was invariably written from right to left.

Addition and subtraction posed few problems. In adding two numbers, one made a collection of each set of symbols that appeared in both numbers, replacing them with the next higher symbol as necessary. Subtraction was merely the reversal of the process for addition, with decomposition achieved by replacing a larger hieroglyph with ten of the next-lower symbol.

The absence of zero is a shortcoming of Egyptian numeration that is often referred to in histories of mathematics. It is clear that an absence of zero as a placeholder is perfectly consistent with a number system such as the Egyptian system. However, in two other senses it may be argued, as Lumpkin (2002, pp. 161–67) has done, that the concept of zero was present in Egyptian mathematics. First, there is zero as a number. Scharff (1922, pp. 58–59) contains a monthly balance sheet of the accounts of a traveling royal party, dating back to around 1770 BC, which shows the expenditure and the income allocated for each type of good in a separate column. The balance of zero, recorded in the case of four goods, is shown by the *nfr* symbol that corresponds to the Egyptian word for "good," "complete," or "beautiful." It is interesting, in this context, that the concept of zero has a positive association in other cultures as well, such as in India *(sunya)* and among the Maya (the shell symbol).

The same *nfr* symbol appears in a series of drawings of some Old Kingdom constructions. For example, in the construction of Meidun Pyramid, it appears as a ground reference point for integral values of cubits given as "above zero" (going up) and "below zero" (going down). There are other examples of these number lines at pyramid sites, known and referred to by Egyptologists early in the century, including Borchardt, Petrie, and Reiner, but not mentioned by historians of mathematics, not even Gillings (1972), who played such an important role in revealing the treasures of Egyptian mathematics to a wider public. About fifteen hundred years after Ahmes, in a deed from Edfu, there is a use of the "zero concept as a replacement to a magnitude in geometry," according to Boyer (1968, p. 18). Perhaps there are other examples waiting to be found in Egypt.

The hieratic representation was similar to the hieroglyphic system in that it was additive and based on powers of 10. But it was far more economical, as a number of identical hieroglyphs were replaced with fewer symbols, or just one symbol. For example, the number 57 was written in hieroglyphic notation as

But the same number would be written in hieratic notation as  where  and  represent 7 and 50 respectively. It is clear that the idea of a ciphered number system, which we discussed in the previous chapter, is already present here.

While the hieratic notation was no doubt more taxing on memory, its economy, speed, conciseness, and greater suitability for writing with pen and ink must have been the main reasons for its fairly early adoption in ancient Egypt. For example, to represent the number 999 would take altogether twenty-seven symbols in hieroglyphics, compared with three number signs in hieratic representation! And from the point of view of the history of mathematics, the hieratic notation may have inspired, at least in its formative stages, the development of the alphabetic Greek number system around the middle of the first millennium BC. Over the years the hieratic was replaced by an "abnormal hieratic"[13] and demotic, while the hieroglyphic script remained in use throughout.

From as early as the First Dynasty in the Archaic period (3000–2686 BC), thin sheets of whitish "paper" were produced from the interior of the stem of a reedlike plant that grew in the swamps along the banks of the Nile. Fresh stems were cut, the hard outer parts removed, and the soft inner pith was laid out and beaten until it formed into sheets, the natural juice of the plant acting as the adhesive. Once dried in the sun, the writing surface was scraped smooth and gummed into rolls, of which the longest known measures over 40 meters. On these rolls the Egyptians wrote with a brushlike pen, using for ink either a black substance made from soot or a red substance made from ocher.

Egyptian Arithmetic

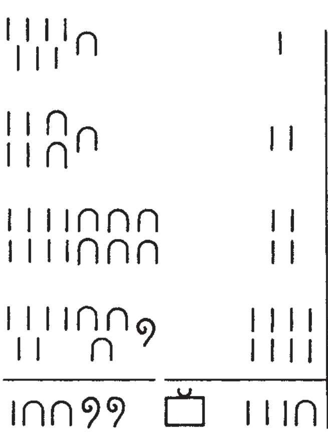
The Method of Duplation and Mediation

One of the great merits of the Egyptian method of multiplication or division is that it requires prior knowledge of only addition and the 2-times table. A few simple examples will illustrate how the Egyptians would have done their multiplication and division. Only in the first example will the operation be explained in terms of both the hieroglyphic and present-day notation.

EXAMPLE 3.1 Multiply 17 by 13.

Solution

The scribe had first to decide which of the two numbers was the multiplicand—the one he would multiply by the other. Suppose he chose 17. He would proceed by successively multiplying 17 by 2 (i.e., continuing to double each result) and stopping before he got to a number on the left-hand side of the "translated" version below that exceeded the multiplier, 13 (= 1 + 4 + 8):



→ 1	17
2	34
→ 4	68
→ 8	136
1 + 4 + 8 = 13	17 + 68 + 136 = 221

The hieroglyph , resembling a papyrus roll, meant "total." The numbers to be added to obtain the multiplier 13 are arrowed.

If this method is to be used for the multiplication of any two integers, the following rule must apply: Every integer can be expressed as the sum of integral powers of 2. Thus

$$15 = 2^0 + 2^1 + 2^2 + 2^3;$$
$$23 = 2^0 + 2^1 + 2^2 + 2^4.$$

It is not known whether the Egyptians were aware of this general rule, though the confidence with which they approached all forms of multiplication by this process suggests that they had an inkling.

This ancient method of multiplication provides the foundation for Egyptian calculation. It was widely used, with some modifications, by the Greeks and continued well into the Middle Ages in Europe. In a modern variation of this method, still popular among rural communities in Russia, Ethiopia, and the Near East, there are no multiplication tables, and the ability to double and halve numbers (and to distinguish odd from even) is all that is required.

EXAMPLE 3.2 Multiply 225 by 17.

Solution

→	225	17
	112	34
	56	68
	28	136
	14	272
→	7	544
→	3	1,088
→	1	2,176

Inspect the left-hand column for odd numbers or "potent" terms (ancient lore in many societies imputed "potency" to odd numbers). Add the corresponding terms in the right-hand column to get the answer.

$$17 + 544 + 1{,}088 + 2{,}176 = 3{,}825.$$

This method, known in the West as the "Russian peasant method," works by expressing the multiplicand, 225, as the sum of integral powers of 2:

$$225 = 1 + 32 + 64 + 128 = 1(2^0) + 0(2^1) + 0(2^2) + 0(2^3) + 0(2^4) + 1(2^5) + 1(2^6) + 1(2^7).$$

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Adding the results of multiplying each of these components by 17 gives

$$(17 \times 2^0) + (17 \times 2^5) + (17 \times 2^6) + (17 \times 2^7) = 17 + 544 + 1{,}088 + 2{,}176 = 3{,}825.$$

In Egyptian arithmetic, the process of division was closely related to the method of multiplication. In the Ahmes Papyrus a division x/y is introduced by the words "reckon with y so as to obtain x." So an Egyptian scribe, rather than thinking of "dividing 696 by 29," would say to himself, "Starting with 29, how many times should I add it to itself to get 696?" The procedure he would set up to solve this problem would be similar to a multiplication exercise:

Example 3.3 Divide 696 by 29.

Solution

1		29
2		58
4		116
8	→	232
16	→	464
16 + 8 = 24		232 + 464 = 696

The scribe would stop at 16, for the next doubling would take him past the dividend, 696. Some quick mental arithmetic on the numbers in the right-hand column shows the sum of 232 and 464 would give the exact value of the dividend 696. Taking the sum of the corresponding numbers in the left-hand column gives the answer 16 + 8 = 24.

Where a scribe was faced with the problem of not being able to get any combination of the numbers in the right-hand column to add up to the value of the dividend, fractions had to be introduced. And here the Egyptians faced constraints arising directly from their system of numeration: their method of writing numerals did not allow any unambiguous way of expressing fractions. But the way they tackled the problem was quite ingenious.

Egyptian Representation of Fractions

Nowadays, we would write a noninteger number as either a fraction (2/7) or a decimal (0.285714). From very early times, the ancient Egyptians (as far as we can tell from surviving documents) wrote nonintegers as a sum of unit fractions. So that while a number like 1/7 in hieroglyphic notation consisted of seven vertical lines crowned by the hieroglyphic sign for mouth, the number 2/7 was written as a sum of unit fractions or 2/7 = 1/4 + 1/28. Note that the same unit fraction was not used twice in one representation (i.e., 2/7 = 1/7 + 1/7 was not allowed); and there is no known explanation for the adoption of this convention. This system of representing fractions became known as the Egyptian system.

There have been different views as to why Egyptians followed this style of fractional representation. A traditional view, widely accepted even today, is that the representation reflected the notational, or conceptual, limitation of their number system. To many of us brought up on fixed-base representations of numbers, it is difficult to imagine a situation that favored the Egyptian usage. One could, of course, contrast the "exactness" of the Egyptian representation with the "approximate" nature of fixed-base expansions of the neighboring Mesopotamians, whose system is discussed in chapter 4. We could marvel at the way that Egyptian scribes performed complicated arithmetical operations. They would take 16 + 1/56 + 1/679 + 1/776, find 2/3 of it as 10 + 2/3 + 1/84 + 1/1,358 + 1/4,074 + 1/1,164, then add 1/2 of it and 1/7 of it and show that it all adds up to 37!

The Egyptian preference for "exactness" has some interesting historical parallels with the preference of their students, the Greeks, for geometry over symbolic arithmetic. The Egyptian style may have contributed to the Pythagorean number mysticism and the number theory that grew out of it. For example, the notion of a "perfect number," which is equal to the sum of its proper divisors, would now seem to many as esoteric. But perfect numbers have certain practical uses in working with Egyptian fractions. And the subject provides, even today, a source of mathematical puzzles and problems in abstract number theory.

The puzzle still remains: what practical purpose was served by the Egyptian unit fractions? One reason for expanding rational fractions is to facilitate easy comparison of different quantities. For example, if you had to choose between being paid 1/7 of a bushel of corn or 13/89 of a bushel, which should you take? In terms of the Babylonian arithmetic the two quantities could be expressed in base 60 as

$$\frac{1}{7} \approx \frac{8}{60} + \frac{34}{60^2} + \frac{17}{60^3},$$

$$\frac{13}{89} \approx \frac{8}{60} + \frac{45}{60^2} + \frac{50}{60^3}.$$

It is clear from inspection that while the first terms on the right of both quantities are identical, the second term of 13/89 is larger than the second term of 1/7.

An estimate of the relative magnitude of the two fractions using the Egyptian approach is cumbersome. For example, the decomposition of 13/89 is impossibly complicated, as shown below:

$$\frac{13}{89} = \frac{(1+4+8)}{89} = \frac{1}{89} + \left(\frac{1}{30} + \frac{1}{178} + \frac{1}{267} + \frac{1}{445}\right) + \left(\frac{1}{15} + \frac{1}{89} + \frac{2}{267} + \frac{2}{445}\right)$$

$$= \frac{1}{15} + \frac{1}{30} + \frac{1}{60} + \frac{1}{178} + \frac{1}{267} + \frac{1}{356} + \frac{1}{445} + \frac{1}{534} + \frac{1}{890} + \frac{2}{267} + \frac{2}{445}.$$

Note that the last two terms in the expansion for 13/89 can be converted to unit fractions if we had a $2/n$ table extending up to $n = 445$. (The $2/n$ table given in the Ahmes Papyrus and shown in table 3.1 later in this chapter provides only odd values of n up to 101*)*. At some point, the quantities have to be reduced to expansions that have a common denominator for a comparison to be made.

Operations with Unit Fractions

Operating with unit fractions is a singular feature of Egyptian mathematics and is absent from almost every other mathematical tradition. A substantial proportion of surviving ancient Egyptian calculations make use of such operations—of the eighty-seven problems in the Ahmes Papyrus, only six do not. Two reasons may be suggested for this great emphasis on fractions. In a society that did not use money, where transactions were carried out in kind, there was a need for accurate calculations with fractions, particularly in practical problems such as division of food, parceling out land, and mixing different ingredients for beer or bread. We shall see later that a number of problems in the Ahmes Papyrus deal with such practical concerns.

A second reason arose from the peculiar character of Egyptian arithmetic. The process of halving in division often led to fractions. Consider how the Egyptians solved the following problem (no. 25) from the Ahmes Papyrus.

EXAMPLE 3.4 Divide 16 by 3.

Solution

1	→	3
2		6
4	→	12
2/3		2
1/3	→	1
$\overline{1} + \overline{4} + \overline{1/3} = \overline{5} + \overline{1/3}$		$\overline{16}$

As 12 + 3 = 15 falls one short of 16, the Egyptian scribe would proceed by working out 2/3 of 1 and then halving the result (i.e., $1/2 \times 2/3 = 1/3$). These steps are shown on the left. Now, 3 + 12 + 1 = 16. The sum of the corresponding figures in the left-hand column gives the answer $5\frac{1}{3}$.

Two important features of Egyptian calculations with fractions are highlighted here:

1. Perverse as it may seem to us today, to calculate a third of a number a scribe would first find two-thirds of that number and then halve the result. This was standard practice in all Egyptian computations.

2. Apart from two-thirds (represented by its own hieroglyph, either  or ), Egyptian mathematics had no compound fractions: all fractions were decomposed into a sum of unit fractions (fractions such as 1/4 and 1/5).

To represent a unit fraction, the Egyptians used the symbol , meaning "part," with the denominator underneath. Thus 1/5 and 1/40 would appear as  and  respectively.

The 2/*n* Table: Its Construction

The dependence on unit fractions in arithmetical operations, together with the peculiar system of multiplication, led to a third aspect of Egyptian computation. Every multiplication and division involving unit fractions would invariably lead to the problem of how to double unit fractions. Now, doubling a unit fraction with an even denominator is a simple matter of halving the denominator. Thus doubling 1/2, 1/4, 1/6, and 1/8 yields 1, 1/2, 1/3, and 1/4. Doubling 1/3 raised no difficulty, for 2/3 had its own hieroglyphic or hieratic symbol. But it was in doubling unit fractions with other odd denominators that difficulties arose. For some reason unknown to us, it was not permissible in Egyptian computation to write 2 times $1/n$ as $1/n + 1/n$. Thus the need arose for some form of ready reckoner that would provide the appropriate unit fractions that summed to $2/n$, where $n = 5, 7, 9, \ldots$.

At the beginning of the Ahmes Papyrus there is a table of decomposition of $2/n$ into unit fractions for all odd values of n from 3 to 101. In the papyrus, the decomposed unit fractions are marked in red ink. A few of its entries are given in table 3.1.

The usefulness of this table for computations cannot be overemphasized: it may quite legitimately be compared in importance to the logarithmic tables that were used before the advent of electronic calculators. The table is interesting for a number of reasons. For one, it does not contain a single arithmetical error, in spite of the long and highly involved calculations that its construction must have entailed; it may be a final corrected version of a number of earlier attempts that have not survived.

There is an even more remarkable aspect to this table. With the help of a computer, it has been worked out that there are about twenty-eight thousand different combinations of unit fraction sums that can be generated for $2/n$, $n = 3, 5, \ldots, 101$. The constructor of this table arrived at a particular subset of fifty unit-fraction expressions, one for each value of n. According to Gillings (1972), it is possible to discern certain guidelines for the sets of values chosen. There is

1. a preference for small denominators, and none greater than 900;
2. a preference for combinations with only a few unit fractions (no expression contains more than four);
3. a preference for even numbers as denominators, especially as the denominator of the first unit-fraction in each expression, even

TABLE 3.1: SOME ENTRIES FROM THE AHMES PAPYRUS 2/*N* TABLE

2/n	*Unit fractions*
2/5	1/3 + 1/15
2/7	1/4 + 1/28
2/9	1/6 + 1/18
2/15	1/10 + 1/30
2/17	1/12 + 1/51 + 1/68
2/19	1/12 + 1/76 + 1/114
2/45	1/30 + 1/90
2/47	1/30 + 1/141 + 1/470
2/49	1/28 + 1/196
2/51	1/34 + 1/102
2/55	1/30 + 1/330
2/57	1/38 + 1/114
2/59	1/36 + 1/236 + 1/531
2/95	1/60 + 1/380 + 1/570
2/97	1/56 + 1/679 + 1/776
2/99	1/66 + 1/198
2/101	1/101 + 1/202 + 1/303 + 1/606

though they are large or might increase the number of terms in the expression.

To take an example, according to Gillings's calculations the fraction 2/17 can be decomposed into unit-fraction summations in just one way if there are two unit-fraction terms, 11 ways with three unit-fraction terms, and 467 ways with four unit-fraction terms. Table 3.1 shows that the constructor opted for one of the three-unit-fraction groups, 2/17 = 1/12 + 1/51 + 1/68, rather than the solitary two-unit-fraction group, 2/17 = 1/9 + 1/153. It would seem that criteria (1) and (3) prevailed in this instance.[14]

Multiplication and Division with Unit Fractions

The main purpose of constructing the table was to use it for multiplication and division. Let us consider one example of each to illustrate its use. First, multiplication.

EXAMPLE 3.5

Multiply $1\frac{8}{15}$ by $30\frac{1}{3}$ (or $1 + \frac{1}{3} + \frac{1}{5}$ by $30 + \frac{1}{3}$).

Solution

	1	1 + 1/3 + 1/5
→	2	2 + 2/3 + 2/5 = 2 + 2/3 + 1/3 + 1/15
→	4	6 + 2/15 = 6 + 1/10 + 1/30
→	8	12 + 1/5 + 1/15
→	16	24 + 2/5 + 2/15 = 24 + 1/3 + 1/15 + 1/10 + 1/30
	2/3	2/3 + 2/9 + 2/15 = 2/3 + 1/6 + 1/18 + 1/10 + 1/30
→	1/3	1/3 + 1/12 + 1/36 + 1/20 + 1/60
	2 + 4 + 8 + 16 + 1/3 = 30 + 1/3	46 + 1/5 + 1/10 + 1/12 + 1/15 + 1/30 + 1/36

The product of the two numbers using modern multiplication would be $46\frac{23}{45}$, which is exactly equivalent to the Egyptian result given in the last row. In the course of multiplication we have taken the unit fraction terms of 2/5, 2/15, and 2/9 from table 3.1. And because Egyptian multiplication was based on doubling, only table 3.1 was required. The sheer labor and tedium of this form of multiplication should not make us forget how modest is the "tool kit" required. The ability to double, halve, and work with the fraction "two-thirds," together with the $2/n$ table, is sufficient.

To illustrate division with fractions, we take one of the more difficult problems of its kind from the Ahmes Papyrus, problem 33, which may be restated in modern language as follows.

EXAMPLE 3.6 The sum of a certain quantity together with its two-thirds, its half, and its one-seventh becomes 37. What is the quantity?

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Solution

In the language of modern algebra, this problem is solved by setting up an equation of the first degree in one unknown. Let the quantity be x. The problem is then to solve

$$\left(1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{7}\right)x = 37$$

to give

$$x = 37 \div \left(1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{7}\right) = 16\frac{2}{97}.$$

The problem restated: Divide 37 by (1 + 2/3 + 1/2 + 1/7).

	1	1 + 2/3 + 1/2 + 1/7
	2	4 + 1/3 + 1/4 + 1/28 (2/7 = 1/4 + 1/28 from the 2/n table)
	4	8 + 2/3 + 1/2 + 1/14
	8	18 + 1/3 + 1/7
→	16	36 + 2/3 + 1/4 + 1/28

At this point in the procedure, two questions arise:

1. The right-hand side of the last row is close to 37, which is the dividend. What must be added to 2/3 + 1/4 + 1/28 to make up 1? With our present method, we find the answer, 1/21.
2. The next question is: By what must the divisor 1 + 2/3 + 1/2 + 1/7 be multiplied to get 1/21? The answer is 2/97, or in unit fractions 1/56 + 1/679 + 1/776, from table 3.1.

So the solution is

$$37 \div \left(1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{7}\right) = 16 + \frac{1}{56} + \frac{1}{679} + \frac{1}{776} = 16\frac{2}{97}.$$

Egyptian Division: The Use of “Red Auxiliaries”

The real question remains: How would the Egyptians, working within the constraints of their arithmetic, have dealt with the problems raised by questions 1 and 2 in example 3.6? A study of some of the problems in the Ahmes Papyrus provides us with the answer. Problems 21 to 23 are commonly known as "problems in completion," since they are expressed as

Complete 2/3 1/15 to 1. (Problem 21)

Complete 1/4 1/8 1/10 1/35 1/45 to 3. (Problem 23)

These problems are similar to the one in question 1 above, which may also be expressed in this way:

Complete 2/3 1/4 1/28 to 1.

The Egyptians adopted a method of solution that is analogous (but not equivalent) to the present-day method of least common denominator. First they took the denominator of the smallest unit-fraction as a reference number, and then they multiplied each of the fractions by this number to obtain "red auxiliaries" (so named because the scribe wrote these numbers in red ink). They proceeded to calculate by how much the sum of these auxiliaries fell short of the reference number. This shortfall quantity was then expressed as a fraction of the reference number to obtain the desired complement. If the shortfall quantity turned out to be an awkward fraction, a further search was made for a reference number that would result in more manageable auxiliaries. So, how was question 1 tackled the Egyptian way?

Example 3.7 Complete 2/3 1/4 1/28 to 1.

Solution

$$\frac{2}{3} + \frac{1}{4} + \frac{1}{28} + \text{(some fraction)} = 1;$$

$$28 + \left(10 + \frac{1}{2}\right) + \left(1 + \frac{1}{2}\right) + 2 = 42.$$

The denominator of the smallest fraction, 28, is not a suitable reference number given the auxiliaries that result. Instead, 42 is chosen for ease of calculation because it is important that the sum of the auxiliaries belonging to the divisor in example 3.6 is an integer. Thus 42 is the lowest common multiple of the numbers 1, 3, 2, and 7. However, the reference number chosen in Egyptian computation was not necessarily the lowest common multiple. So what fraction(s) of 42 will give 2? The answer is 1/21.

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The next step is to find by what fraction the divisor 1 + 2/3 + 1/2 + 1/7 (from example 3.6) must be multiplied to get 1/21. In other words, we have to divide 1/21 by 1 + 2/3 + 1/2 + 1/7:

→	1	21
→	2/3	14
→	1/2	10 + 1/2
→	1/7	3
	1 + 2/3 + 1/2 + 1/7	48 + 1/2

Now, 1 ÷ (48 + 1/2) = 2/97 = 1/56 + 1/679 + 1/776 (obtained from the 2/*n* table).

Hence 37 ÷ (1 + 2/3 + 1/2 + 1/7) = 16 + 1/56 + 1/679 + 1/776.

We have not followed the scribe all the way in his solution to the problem, for the reason that at one stage his approach requires an addition of sixteen unit-fractions, the last six of which are 1/1,164, 1/1,358, 1/1,552, 1/4,074, 1/4,753, and 1/5,432! We can only assume that the scribe was either an incredible calculator or that he had a battery of tables that he could consult when called upon to add different combinations of unit fractions. However, the more likely but mundane explanation is that the scribe "cheated," since he knew what the answer should be! The fact remains: the Egyptians were inveterate table makers, and the summation table of unit fractions contained in the Leather Roll and the decomposition table of 2/*n* in the Ahmes Papyrus are prime examples.[15]

It is unlikely that the original problem (example 3.6) had any practical import. In an attempt probably to illustrate, for the benefit of trainee scribes, the solution of simple equations of this type, an unfortunate choice of numbers led to difficult sets of unit fractions with the attendant cumbersome operations, which the scribe accomplished without faltering. One is again struck by the mental agility of the scribes who could perform such feats with a minimum of mathematical tools to call upon. The use of red auxiliaries is further evidence of the high level of Egyptian achievement in computation, since they enabled any division, however complicated, to be performed.