Applications of Unit Fractions: Distribution of Loaves

As has been suggested, the exclusive use of unit fractions in Egyptian mathematics also had a practical rationale. This is brought out quite clearly in the first six problems of the Ahmes Papyrus, which are concerned with sharing out n loaves among ten men, where $n = 1, 2, 6, 7, 8, 9$. As an illustration let us consider problem 6, which relates to the division of 9 loaves among ten men. A present-day approach would be to work out the share of each man, i.e., 9/10 of a loaf, and then divide the loaves so that the first nine men would each get 9/10 cut from one of the 9 loaves. The last man, however, left with the 9 pieces of 1/10 remaining from each loaf, might well regard this method of distribution as less than satisfactory. The Egyptian method of division avoids such a difficulty. It consists of first looking up the decomposition table for $n/10$ and discovering that $9/10 = 2/3 + 1/5 + 1/30$. The division would then proceed as shown in figure 3.1: seven men would each receive 3 pieces of bread, consisting of 2/3, 1/5, and 1/30 of a loaf. The other three men would each receive 4 pieces consisting of two 1/3 pieces, a single 1/5 piece, and a single 1/30 of a loaf. Justice is not only done, but seen to be done!

Applications of Unit Fractions: Remuneration of Temple Personnel

In a nonmonetary economy, payment for both goods and labor is made in kind. Often the choice of the goods that act as measures or standards

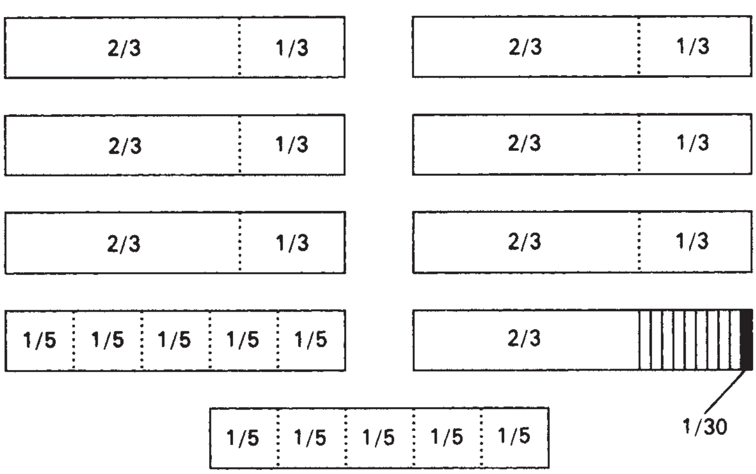

FIGURE 3.1: Problem 6 from the Ahmes Papyrus: sharing 9 loaves among 10 men (After Gillings 1962, p. 67)

of value provides interesting insights into the character of the society. In Egypt, bread and beer were the most common standards of value for exchange. A number of problems in the Ahmes Papyrus concern these goods, dealing with their distribution among a given number of workers, and also with strength (*pesu*) of different types of these two commodities. We shall be examining one of the *pesu* problems later in this chapter. But first we look at an example, brought to our attention by Gillings (1972), that sheds some interesting light.

Table 3.2, adapted from Gillings's book, is a record of payments to various temple personnel at Illahun around 2000 BC. The payments were made in loaves of bread and two different types of beer (referred to here as beer A and beer B). The temple employed 21 persons and had 70 loaves, 35 jugs of beer A, and $115\frac{1}{2}$ jugs of beer B available for distribution every day. The unit of distribution was taken to be 1/42 of a portion of each of these items, which worked out as 1 + 2/3 loaves of bread, 2/3 + 1/6 jug of beer A, and 2 + 2/3 + 1/10 jugs of beer B.

The table is interesting for a number of reasons. It contains an interesting example of an arithmetical error on the part of a scribe: the unit of distribution of beer B was wrongly worked out as 2 + 2/3 + 1/10 (the correct

Table 3.2: Remuneration of the Personnel of Illahun Temple (Units of Distribution per Person)

		Commodity		
Status of personnel	*Number of portions received*	*Bread (1 + 2/3 loaves)*	*Beer A (2/3 + 1/6 jugs)*	*Beer B* (2 + 1/2 + 1/4 jugs)*
Temple director	10	16 + 2/3	8 + 1/3	27 + 1/2
Head reader	6	10	5	16 + 1/2
Usual reader	4	6 + 2/3	3 + 1/3	11
Head lay priest	3	5	2 + 1/2	8 + 1/4
Priests, various (7)	14	23 + 1/3	11 + 1/3	37 + 1/2
Temple scribe	1 + 1/3	2 + 1/6 + 1/18	1 + 1/9	3 + 2/3
Clerk	1	1 + 2/3	2/3 + 1/6	2 + 1/2 + 1/4
Other workers (8)	2 + 2/3	4 + 1/3 + 1/9	2 + 1/6 + 1/18	7 + 1/3
Totals	42	70	35	115 + 1/2

Note: Adapted from table 11.2 in Gillings (1972)

*The scribe made an error in working out the amount in one portion of beer b, 1/42 of 115 + 1/2, which he estimated as 2 + 2/3 + 1/10 instead of 2 + 1/2 + 1/4. This mistake has been rectified along the lines indicated by Gillings.

value is $2 + 1/2 + 1/4$) . The scribe proceeded to use the incorrect figure in working out the shares of different personnel, but did not apparently check his calculations by adding all the shares. He wrote total as $115\frac{1}{2}$ jugs, which is what it should have been, whereas the table adds up to $114\frac{1}{2}$ jugs. Also, the table bears ample testimony to the facility with which the Egyptians could handle fractions. Given all the limitations of their number system, they proved to be extremely adept at computations. Further, the minute fractional division of both beer and bread suggests a highly developed system of weights and measures: it is intriguing how $2 + 1/6 + 1/18$ jugs of beer A were shared equally among eight "other workers."

From the table we have some indication of the relative status of the personnel at the temple. At the top was the high priest, or temple director, often a member of the royal family. Among his duties was to pour out the drink-offering to the gods and to examine the purity of the sacrificial animals. It was only after he had "smelt" the blood and declared it pure that pieces of flesh could be laid on the table of offerings. Hence *Ue'b*, meaning "pure," was the name by which he was known. Perhaps more important than the *Ue'b*, from a ritual point of view, was the head reader (or reciter-priest), whose duty it was to recite from the holy books. Since magical powers were attributed to these texts, it was generally believed that the reciter-priest was a magician, making him in status and remuneration second only to the high priest. After him came other classes of the priesthood, the largest of which was known as the "servants of God." Some of them were prominent in civil life; others were appointed to serve particular gods. Their job included washing and dressing statues of assigned deities and making offerings of food and drink to them at certain times of the day. The scribes came quite low on the list, though this was not the case in other walks of life—most scribes, particularly those associated with the royal court, enjoyed considerable status and power.

Egyptian Algebra: The Beginnings of Rhetorical Algebra

It is sometimes claimed that Egyptian mathematics consisted of little more than applied arithmetic, and that one cannot therefore talk of Egyptian algebra or geometry. We shall come to the question of Egyptian geometry, but first we consider the existence or otherwise of an entity called Egyptian algebra. Algebra may be defined as a branch of mathematics of generalized arithmetical operations, often involving today the substitution of letters for numbers to express mathematical relationships.

The rules devised by mathematicians for solving problems about numbers of one kind or another may be classified into three types. In the early stages of mathematical development these rules were expressed verbally and consisted of detailed instructions, without the use of any mathematical symbols (such as $+$, $-$, $\div$, $\surd$), about what was to be done to obtain the solution to a problem. For this reason this approach is referred to as "rhetorical algebra." In time, the prose form of rhetorical algebra gave way to the use of abbreviations for recurring quantities and operations, heralding the appearance of "syncopated algebra." Traces of such algebra are to be found in the works of the Alexandrian mathematician Diophantus (c. AD 250), but it achieved its fullest development—as we shall see in later chapters—in the work of Indian and Islamic mathematicians during the first millennium AD. During the past five hundred years "symbolic algebra" has developed. In this type of algebra, with the aid of letters and signs of operation and relation ($+$, $-$, $\div$, $\surd$), problems are stated in such a form that the rules of solution may be applied consistently and systematically. The transformation from rhetorical to symbolic algebra marks one of the most important advances in mathematics. It had to await

1. the development of a positional number system, which allowed numbers to be expressed concisely and with which operations could be carried out efficiently;
2. the emergence of administrative and commercial practices which helped to speed the adoption, not only of such a number system, but also of symbols representing operators.

It is taking too narrow a view to equate the term "algebra" just with symbolic algebra. If one examines the hundred-odd problems in the existing Egyptian mathematical texts, of which most are found in the Ahmes Papyrus, one finds that they are framed in a manner that may be described as "rhetorical" and "algorithmic" or procedure-based. Further, in the case of examples from the Ahmes Papyrus, one can discern distinct stages in laying out a problem and its solution: statement of the problem, the procedure for its solution, and verification of the result. It is interesting to note that the examples in the Moscow Papyrus contain just the statement of the problem (or a diagram) and cryptic instructions for its solution.

As an illustration let us look at problem 72 of the Ahmes Papyrus, restated in modern terminology. It should be noted here that since the Egyptian system of rationing involved the two staple commodities of grain and beer, a frequent task set for the scribes was to record and calculate the amounts and types of these commodities that had to be allocated to various employees and beneficiaries.

Example 3.8 100 loaves of *pesu* 10 are to be exchanged for a certain number of loaves of pesu 45. What is this certain number?

(Note: The word *pesu* (or *psw*) may be defined as a measure of the "weakness" of a commodity. Here it can be taken to be the ratio of the number of loaves produced to the amount of grain used in their production so that the higher the *pesu*, the weaker the bread.)

Solution

We would tackle the above problem today as one of simple proportions, obtaining the number of loaves as $45/10 \times 100 = 450$. The solution prescribed in the Egyptian text is quite involved. It is interesting from our point of view because it contains the germs of algebraic reasoning. Below are the Egyptian solution and a restatement of the same steps in modern symbolic terms.

Egyptian Explanation	Modern Explanation
	Let x and y be the loaves of p and q *pesu*, respectively. Find y if x, p, q are known.
1. Find excess of 45 over 10: result 35. Divide this 35 by 10: result 3 + 1/2.	$(q - p)/p$
2. Multiply this (3 + 1/2) by 100: result 350. Add 100 to 350: result 450.	$[(q - p)/p]x + x$
3. Then the exchange is 100 loaves of 10 *pesu* for 450 loaves of 45 *pesu*.	$y = [(q - p)/p]x + x = (q/p)x$

What is important here is not whether the scribe arrived at this method of solution by any thought process akin to ours, but that what we have here from four thousand years ago is a form of algebra, dependent on knowing that $y/x = q/p$ and $(y - x)/x = (q - p)/p$.

Solving Simple and Simultaneous Equations: The Egyptian Approach

To find topics that are represented in modern elementary algebra, we have to turn to problems 24–34 of the Ahmes Papyrus. One of these problems, problem 26, will serve as an illustration.

EXAMPLE 3.9 A quantity, its 1/4 added to it so that 15 results. [I.e., a quantity and its quarter added become 15. What is the quantity?]

Solution

In terms of modern algebra, the solution is straightforward and involves finding the value of x, the unknown quantity, from an equation:

$$x + \frac{1}{4}x = 15, \quad \text{so} \quad x = 12.$$

The scribe, however, reasoned as follows: If the answer were 4, then 1 + 1/4 of 4 would be 5. The number that 5 must be multiplied by to get 15 is 3. If 3 is now multiplied by the assumed answer (which is clearly false), the correct answer will result: $4 \times 3 = 12$.

This problem belongs to a set of problems that are described as "quantity" or "number" problems and are basically concerned with showing how to determine an unknown quantity from a given relationship. The scribe was using the oldest and probably the most popular way of solving linear equations before the emergence of symbolic algebra—the method of false assumption (or false position). Variants of "quantity" problems of this kind included adding a multiple of an unknown quantity instead of a fraction of the unknown quantity. For example, problem 25 of the Moscow Papyrus asks for a method of calculating an unknown quantity such that twice that quantity together with the quantity itself adds up to 9. The instruction for its solution suggests assuming the quantity as 1 and that together with twice the assumed quantity gives 3. The number that 3 must be multiplied by to get 9 is 3. So the unknown quantity is 3. It is interesting to reflect that such an approach was still in common use in Europe and elsewhere until about a hundred years ago.

The Berlin Papyrus contains two problems that would appear to us today to involve second-degree simultaneous equations (i.e., equations with terms like x^2 and xy). It is badly mutilated in places, so the solution offered below is both conjectural and a reconstructed one.

EXAMPLE 3.10 It is said to thee [that] the area of a square of 100 [square cubits] is equal to that of two smaller squares. The side of one is 1/2 + 1/4 of the other. Let me know the sides of the two unknown squares.

Solution 1: The Symbolic Algebraic Approach

Let x and y be the sides of the two smaller squares. From the information given above, we can derive the following set of equations:

$$x^2 + y^2 = 100;$$
$$4x - 3y = 0.$$

The solution set, $x = 6$ and $y = 8$, is obtained by substituting $x = (3/4)y$ into $x^2 + y^2 = 100$.

Solution 2: The Egyptian Rhetorical Algebraic Approach

Take a square of side 1 cubit (i.e., a false value of y equal to 1 cubit). Then the other square will have side 1/2 + 1/4 cubits (i.e., $x = 1/2 + 1/4$). The areas of the squares are 1 and (1/2 + 1/16) square cubits respectively. Adding the areas of the two squares will give 1 + 1/2 + 1/16 square cubits. Take the square root of this sum: 1 + 1/4. Take the square root of 100 square cubits: 10. Divide 10 by 1 + 1/4. This gives 8 cubits, the side of one square. (So from the false assumption $y = 1$, we have deduced that $y = 8$.) At this point, the papyrus is so badly damaged that the rest of the solution has to he reconstructed. One can only assume that the side of the smaller square was calculated as 1/2 + 1/4 of the side of the larger square, which was 8 cubits. So the side of the smaller square is 6 cubits.

Geometric and Arithmetic Series

A series is the sum of a sequence of terms. The most common types are the arithmetic and geometric series. The terms of the former are an arithmetic progression, a sequence in which each term after the first (usually denoted by a) is obtained by adding a fixed number, called the common difference (usually denoted by d), to the preceding term. For example, 1, 3, 5, 7, 9, . . . is an arithmetic progression with $a = 1$ and $d = 2$. In a geometric progression, each term after the first (a) is formed from the preceding term by multiplying by a fixed number called the common ratio (usually denoted by r). For example, 1, 2, 4, 8, 16, . . . is a geometric progression with $a = 1$ and $r = 2$.

The Egyptian method of multiplication leads naturally to an interest in such series, since it is based on operations with the basic geometric progression 1, 2, 4, 8, . . . and an understanding that any multiplier may be expressed as the sum of elements of this sequence. It would follow that Egyptian interest would focus on finding rules that made it easier to add up certain elements of such sequences. Here is problem 79 from the Ahmes Papyrus.

Example 3.11 The actual statement of the problem in the Ahmes Papyrus is uncharacteristically ambiguous. It presents the following information, and nothing else:

Houses	7
Cats	49
Mice	343
Emmer wheat	2,401[16]
Hekats of grain	16,807
Total	19,607

1	2,801
2	5,602
4	11,204
Total	19,607

There is no algorithm nor any instruction for solution, except for what can be inferred from the calculations. The presentation of this curious data has led to some interesting suggestions. It was first believed that the problem was merely a statement of the first five powers of 7, along with their sum; and that the words "houses," "cats," and so on were really a symbolic terminology for the second and third powers, and so on. Since no such terminology occurs elsewhere, this explanation is unconvincing. Moreover, it does not account for the other set of data on the right.

A more plausible interpretation is that we have here an example of a geometric series, where the first term (a) and common ratio (r) are both 7, which shows that the sum of the first five terms of the series is obtained as $7[1 + (7 + 49 + 343 + 2{,}401)] = 7 \times 2{,}801$. We now see that the second set of data in the problem is merely the multiplication of 7 by 2,801 in the Egyptian way.

A precursor to this Egyptian example of geometric progression may have been a mathematical text from the Old Babylonian period dealing with the same subject. It was discovered at Mari, a small kingdom in the northwest corner of Mesopotamia, which was conquered by Hammurabi in 1757 BC. A reconstruction of this example, which Friberg (2005, p. 5) describes as a "whimsical story," reads as follows: "There were 645,539 barleycorns, 9 barleycorns on each ear of barley, 9 ears of barley eaten by each ant, 9 ants swallowed by each bird, 9 birds caught by each of 99 men. How many were there altogether?" [Answer: 730,719 different items.]

In the next chapter, in the section dealing with geometric series in Mesopotamian mathematics, we return to this problem. But what is being strongly suggested here is the existence of links between the two mathematical traditions, long considered to have been independent of one another. We will return to this theme in chapter 5.

A detailed solution to another problem in the Ahmes Papyrus gives some support to the view that the Egyptians had an intuitive rule for summing n terms of an arithmetic progression. Problem 64 may be restated as follows:

EXAMPLE 3.12 Divide 10 hekats of barley among 10 men so that the common difference is one-eighth of a hekat of barley.

Solution

The solution of the problem as it appears in the Papyrus is given on the left-hand side. On the right-hand side the algorithm is stated symbolically.

EGYPTIAN METHOD	SYMBOLIC EXPRESSION
	Let a be the first term, f the last term, d the difference, n the number of terms, and S the sum of n terms.
1. Average value: 10/10 = 1.	1. Average value of n terms $= S/n$.
2. Total number of common differences: $10 - 1 = 9$.	2. Number of common differences $= n - 1$

Continued . . .

Continued . . .

3. Find half the common difference: $1/2 \times 1/8 = 1/16$.	3. Half the common difference $= d/2$.
4. Multiply 9 by 1/16: $1/2 + 1/16$.	4. Multiply $n - 1$ by $d/2$: $(n - 1)d/2$.
5. Add this to the average value to get the largest share: $1 + 1/2 + 1/16$.	5. $f = S/n + (n - 1)d/2$.
6. Subtract the common difference (1/8) nine times to get the lowest share: $1/4 + 1/8 + 1/16$.	6. $a = f - (n - 1)d$.
7. Other shares are obtained by adding the common difference to each successive share, starting with the lowest. The total is 10 *hekats* of barley.	7. Now form $a, a + d, a + 2d \ldots, a + (n - 1)d$. So $S = an + (1/2)\, n(n - 1)d$, or $S/n = a + (1/2)\,(n - 1)d$.

The correspondence between the rhetorical algebra of the Egyptians and our symbolic algebra is quite close, though a word of caution is necessary here. It would not be reasonable to infer, on the basis of this correspondence, that the ancient Egyptians used anything like the algebraic reasoning on the right-hand side. It is more likely that they took a common-sense approach, listing the following sequence on the basis that the terms added to 10:

$$a, a + \frac{1}{8}, a + \frac{2}{8}, \ldots, a + \frac{9}{8}.$$

Each successive term gives the rising share of barley received by the 10 men.

Egyptian Geometry

The practical character of Egyptian geometry has led a number of commentators to question whether it can properly be described as geometry,