Chapter 1

"I Vote for Euclid"

In 1864, the Reverend J. P. Gulliver, of Norwich, Connecticut, recalled a conversation with Abraham Lincoln about how the president had acquired his famously persuasive rhetorical skill. The source, Lincoln said, was geometry.

> In the course of my law-reading I constantly came upon the word *demonstrate*. I thought, at first, that I understood its meaning, but soon became satisfied that I did not. . . . I consulted Webster's Dictionary. That told of "certain proof," "proof beyond the possibility of doubt;" but I could form no idea what sort of proof that was. I thought a great many things were proved beyond a possibility of doubt, without recourse to any such extraordinary process of reasoning as I understood "demonstration" to be. I consulted all the dictionaries and books of reference I could find, but with no better results. You might as well have defined *blue* to a blind man. At last I said, "Lincoln, you can never make a lawyer if you do not understand what *demonstrate* means;" and I left my situation in Springfield, went home to my father's house, and staid there till I could give any propositions in the six books of Euclid at sight. I then found out what "demonstrate" means, and went back to my law studies.

Gulliver was fully on board, replying, "No man can talk well unless he is able first of all to define to himself what he is talking about. Euclid, well studied, would free the world of half its calamities, by banishing half the nonsense which now deludes and curses it. I have often thought that Euclid would be one of the best books to put on the catalogue of the Tract Society, if they could only get people to read it. It would be a means of grace." Lincoln, Gulliver tells us, laughed and agreed: "I vote for Euclid."

Lincoln, like the shipwrecked John Newton, had taken up Euclid as a source of solace at a rough time in his life; in the 1850s, after a single term in the House of Representatives, he seemed finished in politics and was trying to make a living as an ordinary traveling lawyer. He had learned the rudiments of geometry in his earlier job as a surveyor and now aimed to fill the gaps. His law partner William Herndon, who often had to share a bed with Lincoln at small country inns in their sojourns around the circuit, recalls Lincoln's method of study; Herndon would fall asleep, while Lincoln, his long legs hanging over the edge of the bed, would stay up late into the night with a candle lit, deep in Euclid.

One morning, Herndon came upon Lincoln in their offices in a state of mental disarray:

> He was sitting at the table and spread out before him lay a quantity of blank paper, large heavy sheets, a compass, a rule, numerous pencils, several bottles of ink of various colors, and a profusion of stationery and writing appliances generally. He had evidently been struggling with a calculation of some magnitude, for scattered about were sheet after sheet of paper covered with an unusual array of figures. He was so deeply absorbed in study he scarcely looked up when I entered.

Only later in the day did Lincoln finally get up from his desk and tell Herndon that he had been trying to square the circle. That is, he was trying to construct a square with the same area as a given circle, where to "construct" something, in proper Euclidean style, is to draw it on the page using just two tools, a straightedge and a compass. He worked at the problem for two straight days, Herndon remembers, "almost to the point of exhaustion."

> I have been told that the so-called squaring of the circle is a practical impossibility, but I was not aware of it then, and I doubt if Lincoln was. His attempt to establish the proposition having ended in failure, we, in the office, suspected that he was more or less sensitive about it and were therefore discreet enough to avoid referring to it.

Squaring the circle is a very old problem, whose fearsome reputation I suspect Lincoln might actually have known; "squaring the circle" has been a metaphor for a difficult or impossible task for a long time. Dante name-checks it in the *Paradiso*: "Like the geometer who gives his all trying to square the circle, and still can't find the idea he needs, *that's* how I was." In Greece, where it all started, a standard exasperated comment when someone is making a task harder than necessary is to say, "I wasn't asking you to square the circle!"

There is no *reason* one needs to square a circle—the problem's difficulty and fame is its own motivation. People with a conquering mentality tried to square circles from antiquity until 1882, when Ferdinand von Lindemann proved it couldn't be done (and even then a few diehards persisted; okay, even *now*). The seventeenth-century political philosopher Thomas Hobbes, a man whose confidence in his own mental powers is not fully captured by the prefix "over," thought he'd cracked it. Per his biographer John Aubrey, Hobbes discovered geometry in middle age and quite by accident:

> Being in a Gentleman's Library, Euclid's *Elements* lay open, and 'twas the 47 *El. Libri* 1. He read the Proposition. By G_, saydd he (he would now and then sweare an emphaticall Oath by way of emphasis) *this is impossible!* So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. *Et sic deinceps* that at last he was demonstratively convinced of that trueth. This made him in love with Geometry.

Hobbes was constantly publishing new attempts and getting in petty feuds with the major British mathematicians of the time. At one point, a correspondent pointed out that one of his constructions was not quite correct because two points P and Q he claimed to be equal were actually at very slightly different distances from a third point R; 41 and about 41.012 respectively. Hobbes retorted that his points were big enough in extent to cover such a minor difference. He went to his grave still telling people he'd squared the circle.*

An anonymous commentator in 1833, reviewing a geometry textbook, described the typical circle-squarer in a way that quite precisely depicts both Hobbes, two centuries prior, and intellectual pathologies still hanging around here in the twenty-first:

> [A]ll they know of geometry is, that there are in it some things which those who have studied it most have long confessed themselves unable to do. Hearing that the authority of knowledge bears too great a sway over the minds of men, they propose to counterbalance it by that of ignorance: and if it should chance that any person acquainted with the subject has better employment than hearing them unfold hidden truths, he is a bigot, a smotherer of the light of truth, and so forth.

In Lincoln, we find a more appealing character: enough ambition to try, enough humility to accept that he hadn't succeeded.

What Lincoln took from Euclid was the idea that, if you were careful, you could erect a tall, rock-solid building of belief and agreement by rigorous deductive steps, story by story, on a foundation of axioms no one could doubt: or, if you like, truths one holds to be self-evident. Whoever *doesn't* hold those truths to be self-evident is excluded from discussion. I hear the echoes of Euclid in Lincoln's most famous speech, the Gettysburg Address, where he characterizes the United States as "dedicated to the proposition that all men are created equal." A "propo-

* The long and frankly hilarious story of Hobbes's war against his patient mathematical critics is told in chapter 7 of *Infinitesimal*, by Amir Alexander.

sition" is the term Euclid uses for a fact that follows logically from the self-evident axioms, one you simply cannot rationally deny.

Lincoln wasn't the first American to look for a basis of democratic politics in Euclidean terms; that was the math-loving Thomas Jefferson. Lincoln wrote, in a letter read at an 1859 Jefferson commemoration in Boston he was unable to attend:

> One would start with great confidence that he could convince any sane child that the simpler propositions of Euclid are true; but, nevertheless, he would fail, utterly, with one who should deny the definitions and axioms. The principles of Jefferson are the definitions and axioms of free society.

Jefferson had studied Euclid at William and Mary as a young man, and esteemed geometry highly ever afterward.* While vice president, Jefferson took the time to answer a letter from a Virginia student about his proposed plan of academic study, saying: "Trigonometry, so far as this, is most valuable to every man, there is scarcely a day in which he will not resort to it for some of the purposes of common life" (though he describes much of higher mathematics as "but a luxury; a delicious luxury indeed; but not to be indulged in by one who is to have a profession to follow for his subsistence").

In 1812, retired from politics, Jefferson wrote to his predecessor in the presidency, John Adams:

> I have given up newspapers in exchange for Tacitus and Thucydides, for Newton and Euclid; and I find myself much the happier.

Here we see a real difference between the two geometer-presidents. For Jefferson, Euclid was part of the classical education required of a cultivated patrician, of a piece with the Greek and Roman historians and the scientists of the Enlightenment. Not so for Lincoln, the self-educated

* Though "we hold these truths to be self-evident" wasn't Jefferson's line; his first draft of the Declaration has "we hold these truths to be sacred & undeniable." It was Ben Franklin who scratched out those words and wrote "self-evident" instead, making the document a little less biblical, a little more Euclidean.

rustic. Here's the Reverend Gulliver again, recalling Lincoln recalling his childhood:

> I can remember going to my little bedroom, after hearing the neighbors talk of an evening with my father, and spending no small part of the night walking up and down, and trying to make out what was the exact meaning of some of their, to me, dark sayings. I could not sleep, though I often tried to, when I got on such a hunt after an idea, until I had caught it; and when I thought I had got it, I was not satisfied until I had repeated it over and over, until I had put it in language plain enough, as I thought, for any boy I knew to comprehend. This was a kind of passion with me, and it has stuck by me, for I am never easy now, when I am handling a thought, till I have bounded it north and bounded it south, and bounded it east and bounded it west. Perhaps that accounts for the characteristic you observe in my speeches.

This is not geometry, but it's the mental habit of the geometer. You don't settle for leaving things half-understood; you boil down your thoughts and trace back their steps of reason, just as Hobbes had amazedly watched Euclid do. This kind of systematic self-perception, Lincoln thought, was the only way out of confusion and darkness.

For Lincoln, unlike Jefferson, the Euclidean style isn't something belonging to the gentleman or the possessor of a formal education, because Lincoln was neither. It's a hand-hewn log cabin of the mind. Built properly, it can withstand any challenge. And anybody, in the country Lincoln conceived, can have one.

FROZEN FORMALITY

The Lincolnian vision of geometry for the American masses, like a lot of his good ideas, was only incompletely realized. By the middle of the nineteenth century, geometry had moved from college to the public high school; but the typical course used Euclid as a kind of museum piece, whose proofs were to be memorized, recited, and to some extent appreciated. How anyone might have *come up* with those proofs was not to be spoken of. The proof-maker himself almost disappeared: one writer of the time remarked that "many a youth reads six books of the *Elements* before he happens to be informed that Euclid is not the name of a science, but of a man who wrote upon it." The paradox of education: what we most admire we put in a box and make dull.

To be fair, there is not much to say about the historical Euclid, because there is not much we know about the historical Euclid. He lived and worked in the great city of Alexandria, in North Africa, sometime around 300 BCE. That's it—that's what we know. His *Elements* collects the knowledge of geometry possessed by Greek mathematics at the time, and lays the foundations of number theory for dessert. Much of the material was known to mathematicians prior to Euclid's time, but what's radically new, and was instantly revolutionary, is the *organization* of that huge body of knowledge. From a small set of axioms, which were almost impossible to doubt,* one derives step by step the whole apparatus of theorems about triangles, lines, angles, and circles. Before Euclid—if there actually was a Euclid, and not a shadowy collective of geometry-minded Alexandrians writing under that name—such a structure would have been unimaginable. Afterward, it was a model for everything admirable about knowledge and thought.

There is, of course, another way to teach geometry, which emphasizes invention and tries to put the student in the Euclidean cockpit, with the power to make their own definitions and see what comes of them. One such textbook, *Inventional Geometry,* starts from the premise that "the only true education is self-education." Don't look at other people's constructions, the book counsels, "at least until you have discovered a construction of your own," and avoid anxiety and comparing yourself with other students, because everyone learns at their own pace and you're more likely to master the material if you're enjoying yourself. The book itself is no more than a series of puzzles and problems, 446 in all. Some of these are straightforward: "Can you make three angles with

* Except one; but the vexed question of the "parallel postulate," and the two-thousand-year journey toward non-Euclidean geometry it launched, is well-told elsewhere and will only be glanced at here.

two lines? Can you make four angles with two lines? Can you make more than four angles with two lines?" Some of them, the author warns, are not actually solvable, the better to put yourself in the position of a *true* scientist. And some of them, like the very first one, have no clear "right answer" at all: "Place a cube with one face flat on a table, and with another face towards you, and say which dimension you consider to be the thickness, which the breadth, and which the length." Altogether, it is just the kind of "child-centered," exploratory approach that traditionalists deride as what's wrong with education nowadays. It came out in 1860.

A few years ago, the mathematics library at the University of Wisconsin came into possession of a huge trove of old math textbooks, books that had actually been used by Wisconsin schoolchildren over the last hundred years or so* and eventually discarded in favor of newer models. Looking at the weathered books, you see that every controversy in education has been waged before, multiple times, and everything we think of as new and strange—math books like *Inventional Geometry* that ask students to come up with proofs on their own, math books that make problems "relevant" by relating them to students' everyday lives, math books designed to advance social causes, progressive or otherwise—is also old, and was thought of as strange at the time, and no doubt will be new and strange again in the future.

A note in passing: the introduction to *Inventional Geometry* mentions that geometry has "a place in the education of all, not excepting that of women"—the book's author, William George Spencer, was an early advocate of coeducation. A more common nineteenth-century attitude toward women and geometry is conveyed in (but not endorsed by) *The Mill on the Floss*, by George Eliot†, published the same year as Spencer's textbook: "Girls can't do Euclid, can they, sir?" one character asks the schoolmaster Mr. Stelling, who responds, "They've a great deal of superficial cleverness; but they couldn't go far into anything." Stelling

* In one of the books of basic arithmetic, last used around 1930, I found a small penciled notation in the margin: "turn to p. 170"—on p. 170 there was another instruction, "turn to p. 36," where I got a new command, and so on and so on, until I came to the last page, where I found written, "You're a fool!" Pranked by a ten-year-old from beyond the grave.

† In this context it's relevant that "George Eliot" was a pen name for Mary Ann Evans.

represents, in satirically exaggerated form, the traditional mode of British pedagogy Spencer was rebelling against: a long march through memorization of the masters, in which the slow messy process of building understanding is not just neglected but actively guarded against. "Mr. Stelling was not the man to enfeeble and emasculate his pupil's mind by simplifying and explaining." Euclid, a kind of tonic of manliness, was to be suffered straight, like a strong drink or an ice-cold shower.

Even in the highest reaches of mathematical research, dissatisfaction with Stellingism had begun to build. The British mathematician James Joseph Sylvester, whose geometry and algebra (and distaste for the stultified deadness of British academia) we'll be talking about later, thought Euclid should be hidden "far out of the schoolboy's reach," and geometry taught through its relation to physical science, with an emphasis on the geometry of *motion* supplementing Euclid's static forms. "It is this living interest in the subject," Sylvester wrote, "which is so wanting in our traditional and mediaeval modes of teaching. In France, Germany, and Italy, everywhere where I have been on the Continent, mind acts direct on mind in a manner unknown to the frozen formality of our academic institutions."

BEHOLD!

We don't make students memorize and recite Euclid anymore. In the late nineteenth century, textbooks started including exercises, asking students to construct their own proofs of geometric propositions. In 1893, the Committee of Ten, an educational plenum convened by Harvard president Charles Eliot and charged with rationalizing and standardizing American high school education, codified this shift. The point of geometry in high school, they said, was to train up the student's mind in the habits of strict deductive reasoning. This idea has stuck. A survey conducted in 1950 asked five hundred American high school teachers about their objectives in teaching geometry: the most popular answer by far was "To develop the habit of clear thinking and precise expression," which got almost twice as many votes as "To give a knowledge of the facts and principles of geometry." In other words, we are not here to stuff our students with every known fact about triangles, but to develop in them the mental discipline to build up those facts from first principles. A school for little Lincolns.

And what is that mental discipline for? Is it because, at some point in the student's later life, they will be called upon to demonstrate, finally and incontrovertibly, that the sum of the exterior angles of a polygon is 360 degrees?

I keep waiting for that to happen to me and it never has.

The ultimate reason for teaching kids to write a proof is not that the world is full of proofs. It's that the world is full of *non-proofs*, and grown-ups need to know the difference. It's hard to settle for a non-proof once you've really familiarized yourself with the genuine article.

Lincoln knew the difference. His friend and fellow lawyer Henry Clay Whitney recalled: "[M]any a time have I seen him tear the mask off from a fallacy and shame both the fallacy and its author." We encounter non-proofs in proofy clothing all the time, and unless we've made ourselves especially attentive, they often get by our defenses. There are tells you can look for. In math, when an author starts a sentence with "Clearly," what they are really saying is "This seems clear to me and I probably should have checked it, but I got a little confused, so I settled for just asserting that it was clear." The newspaper pundit's analogue is the sentence starting "Surely, we can all agree." Whenever you see this, you should at all costs *not* be sure that all agree on what follows. You are being asked to treat something as an axiom, and if there's one thing we can learn from the history of geometry, it's that you shouldn't admit a new axiom into your book until it really proves its worth.

Always be skeptical when someone tells you they're "just being logical." If they are talking about an economic policy or a culture figure whose behavior they deplore or a relationship concession they want you to make, and not a congruence of triangles, they are not "just being logical," because they're operating in a context where logical deduction—if it applies at all—can't be untangled from everything else. They want you to mistake an assertively expressed chain of opinions as the proof of a theorem. But once you've experienced the sharp *click* of an honest-to-goodness proof, you'll never fall for this again. Tell your "logical" opponent to go square a circle.

What was distinctive about Lincoln, Whitney says, wasn't that he possessed a superpowered intellect. Lots of people in public life, Whitney writes ruefully, are very smart, and among them one finds both the good and the bad. No: what made Lincoln special was that "it was morally impossible for Lincoln to argue dishonestly; he could no more do it than he could steal; it was the same thing to him in essence, to despoil a man of his property by larceny, or by illogical or flagitious reasoning." What Lincoln had taken from Euclid (or what, already existing in Lincoln, harmonized with what he found in Euclid) was *integrity*, the principle that one does not say a thing unless one has justified, fair and square, that one has the right to say it. Geometry is a form of honesty. They might have called him Geometrical Abe.

The one place I'll part ways with Lincoln is in his shaming the author of the fallacy. Because the hardest person to be honest with is yourself, and it's our self-authored fallacies we need to spend the most time and effort unmasking. You should always be prodding your beliefs as you would a loose tooth, or, better, a tooth whose looseness you're not quite sure about. And if something's not solid, shame is not required, just a calm retreat to the ground you're sure about, and a reassessment of where you can get to from there.

That, ideally, is what geometry has to teach us. But the "frozen formality" Sylvester complained about is far from gone. In practice, the lesson we often teach kids in geometry class is, as math writer–cartoonist–raconteur Ben Orlin puts it:

> A proof is an incomprehensible demonstration of a fact that you already knew.

Orlin's example of such a proof is the "right angle congruence theorem," the assertion that any two right angles are congruent to each other. What might be asked of a ninth grader presented with this assertion? The most typical format is the *two-column proof*, a mainstay of geometry education for more than a century, which in this case would look something like this:

angle 1 and angle 2 are both right angles	**given**
the measure of angle 1 equals 90 degrees	**definition of right angle**
the measure of angle 2 equals 90 degrees	**definition of right angle**
the measure of angle 1 equals the measure of angle 2	**transitivity of equality**
angle 1 is congruent to angle 2	**definition of congruence**

"Transitivity of equality" is one of Euclid's "common notions," arithmetic principles he states at the beginning of the *Elements* and treats as prior even to the geometric axioms. It is the principle that two things which are equal to the same thing are thereby equal to each other.*

I don't want to deny that there's a certain satisfaction in reducing everything to such tiny, precise steps. They snap together so satisfyingly, like Lego! That feeling is something a teacher truly wants to convey.

And yet . . . isn't it *obvious* that two right angles are the same thing, just placed on the page in a different place and pointing in a different direction? Indeed, Euclid makes the equality of any two right angles the fourth of his axioms, the basic rules of the game that are taken to be true without proof and from which all else is derived. So why would a modern high school require students to manufacture a proof of this fact when even Euclid said, "Come on, that's obvious?" Because there are many different sets of starting axioms from which one can derive plane geometry, and proceeding exactly as Euclid did is generally no longer considered the most rigorous or the most pedagogically beneficial choice. David Hilbert rewrote the whole foundation from scratch in 1899, and the axioms used in American schools today typically owe more to those laid down by George Birkhoff in 1932.

Whether it's an axiom or not, the fact that two right angles are equal is something the student just plain knows. You can't blame someone for

* Tony Kushner's screenplay for Steven Spielberg's *Lincoln* movie has Lincoln invoke this in a dramatic moment.

being frustrated when you tell them, "You *think* you knew that, but you didn't *really* know it until you followed the steps in the two-column proof." It's a little insulting!

Too much of geometry class is devoted to proving the obvious. I remember well a course in topology I took my first year of college. The professor, a very distinguished elder researcher, spent two weeks proving the following fact: if you draw a closed curve in the plane, no matter how squiggly and weird it may be, the curve cuts the plane into two pieces; the part outside the curve and the part inside.

Now, on the one hand it's quite difficult, it turns out, to write a formal proof of this fact, known as the Jordan Curve Theorem.* On the other hand, I spent those two weeks in a state of barely controlled irritation. Was *this* what math was truly about? Making the obvious laborious? Reader, I zoned out. So did my classmates, among them many future mathematicians and scientists. Two kids who sat right in front of me, very serious students who would go on to earn PhDs in math at top-five universities, would start vigorously making out every time Distinguished Elder Researcher turned back to the board to chalk out yet another delicate argument on a perturbation of a polygon. I mean just really going at it, as if the force of their teen hunger for each other could somehow rip them into another part of the continuum where this proof was not still taking place.

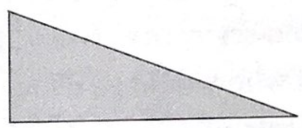
A highly trained mathematician such as my current self might say, standing up a little straighter: well, young people, you are simply not sophisticated enough to know which statements are truly obvious and which conceal subtleties. Perhaps I would bring up the feared Alexander Horned Sphere, which shows that the analogous question in three-dimensional space is not as simple as one might imagine.

But pedagogically, I think that's a pretty bad answer. If we take our time in class to prove things that seem obvious, and insist that those statements are *not* obvious, our students will stew in resentment, just like I did, or find something more interesting to do while the teacher isn't looking.

* Different Jordan.

I like the way master teacher Ben Blum-Smith describes the problem: for students to really feel the fire of math, they have to experience the *gradient of confidence*—the feeling of moving from something obvious to something not-obvious, pushed uphill by the motor of formal logic. Otherwise, we're saying, "Here is a list of axioms that seem pretty obviously correct; put these together until you have another statement that seems pretty obviously correct." It's like teaching somebody about Lego by showing them how you can make two little bricks into one big brick. You can do that, and sometimes you need to, but it is definitely not the point of Lego.

The gradient of confidence is perhaps better experienced than just talked about. If you want to *feel* it, think for a moment about a right triangle.



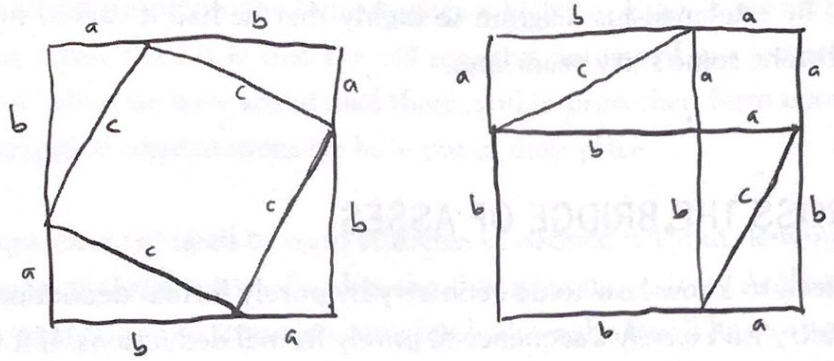
One starts with an intuition: if the vertical and horizontal sides are determined, so is the diagonal side. Walking 3 km south and then 4 km east leaves you a certain distance from your starting point; there is no ambiguity about it.

But what *is* the distance? That's what the Pythagorean Theorem, the first real theorem ever proved in geometry, is for. It tells you that if a and b are the vertical and horizontal sides of a right triangle, and c is the diagonal side, the so-called hypotenuse, then

$$a^2 + b^2 = c^2$$

In case a is 3 and b is 4, this tells us that c^2 is $3^2 + 4^2$, or 9 + 16, or 25. And we know what number, when squared, yields 25; it is 5. That's the length of the hypotenuse.

Why would such a formula be true? You could start climbing the gradient of confidence by literally drawing a triangle with sides 3 and 4 and measuring its hypotenuse—it would look really close to 5. Then draw a triangle with sides 1 and 3 and measure *its* hypotenuse; if you were careful enough with the ruler, you'd get a length really close to 3.16 . . . whose square is 1 + 9 = 10. Increased confidence derived from examples isn't a proof. But this is:



The big square is the same in both pictures. But it's cut up in two different ways. In the first picture, you have four copies of our right triangle, and a square whose side has length c. In the second picture, you also have four copies of the triangle, but they're arranged differently; what's left of the square is now two smaller squares, one whose side has length a and one whose side has length b. The area that remains when you take four copies of the triangle out of the big square has to be the same in both pictures, which means that c^2 (the area left over in the first picture) has to be the same as $a^2 + b^2$ (the area left over in the second).

If we are to be persnickety, we might complain that we have not exactly *proved* that the figure in the first picture is actually a square (that its sides are all the same length is not enough; squeeze opposite corners of a square between your thumb and forefinger and you get a diamond shape called a *rhombus* that's definitely not a square but still has all four sides the same length). But come on. Before you see the picture, you have no reason to think the Pythagorean Theorem is true; after you see it, you know *why* it's true. Proofs like this, where a geometric figure is cut up and rearranged, are called *dissection proofs*, and are prized for their clarity and ingenuity. The twelfth-century mathematician-astronomer Bhāskara* presents a proof of Pythagoras in this form, and finds the picture a demonstration so convincing as not to require verbal explanation, merely a caption that reads "Behold!"† The amateur mathematician Henry Perigal came up with his own dissection proof of Pythagoras in 1830, while trying to square the circle, like Lincoln; he esteemed his diagram so highly that he had it carved into his tombstone some sixty years later.

ACROSS THE BRIDGE OF ASSES

We need to know how to do geometry by purely formal deduction; but geometry isn't *merely* a sequence of purely formal deductions. If it were, it would be no better a way to teach the art of systematic reasoning than a thousand other things. We could teach chess problems, or Sudoku. Or we could make up a system of axioms with no relation to any known human practice at all and force our students to derive their consequences. We teach geometry instead of any of those things because geometry is a formal system that's not *just* a formal system. It's built into the way we think about space, location, and motion. We can't help being geometric. We have, in other words, intuition.

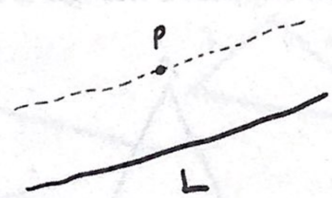
The geometer Henri Poincaré, in a 1905 essay, identifies intuition and logic as the two indispensable pillars of mathematical thought. Every mathematician leans in one direction or the other, and it is the intuition-leaners, Poincaré says, that we tend to call "geometers." We need both pillars. Without logic, we'd be helpless to say anything about a thousand-sided polygon, an object we cannot in any meaningful sense imagine. But without intuition, the subject loses all its savor. Euclid, Poincaré explains, is a dead sponge:

* Often known in mathematical histories as Bhāskara II, to distinguish him from an earlier mathematician with the same name.

† Some sources believe Bhāskara's proof of the Pythagorean Theorem to have been taken from an earlier Chinese source, the *Zhoubi suanjing*, but this is controversial; for that matter, so is the claim that the Pythagoreans themselves had anything we'd now call a proof.

> You have doubtless seen those delicate assemblages of silicious needles which form the skeleton of certain sponges. When the organic matter has disappeared, there remains only a frail and elegant lace-work. True, nothing is there except silica, but what is interesting is the form this silica has taken, and we could not understand it if we did not know the living sponge which has given it precisely this form. Thus it is that the old intuitive notions of our fathers, even when we have abandoned them, still imprint their form upon the logical constructions we have put in their place.

Somehow we need to train students to deduce without denying the presence of the intuitive faculty, the living spongy tissue. And yet we don't want to let our intuition completely drive the bus. The story of the parallel postulate is instructive here. Euclid, as one of his five axioms, listed this one: "Given any line L and any point P not on L, there is one and only one line through P parallel to L."*



This is complicated and chunky compared to his other axioms, which are sleeker things like "any two points are connected by a line." It would be nicer, people thought, if the fifth axiom could be proven from the other four, which felt somehow more primal.

But why? Our intuition, after all, shouts out loud that the fifth axiom is true. What could possibly be more useless than trying to prove it? It's like asking whether we can really prove that 2 + 2 = 4. We *know* that!

And yet mathematicians persisted, trying and failing and trying and failing to show that the fifth axiom followed from the others. And finally they showed that they'd been doomed to fail from the start;

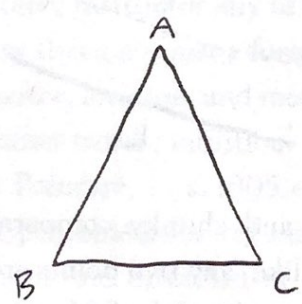
* This isn't exactly the way Euclid formulated it, but it's equivalent to his fifth axiom, which he phrased in an even chunkier and more complicated way.

because there were *other* geometries, in which "line" and "point" and "plane" meant something other than what Euclid (and probably you) mean by those words, but that nonetheless satisfied the first four axioms while failing the last. In some of these geometries, there were infinitely many lines through P parallel to L. In some, there were none.

Isn't that cheating? We weren't asking about *other* bizarro-world geometric entities which we perversely refer to as "lines." We were talking about *actual lines,* for which Euclid's fifth is certainly true.

Sure, that's a tack you're free to take. But by doing so, you're willfully closing off access to a whole world of geometries, just because they're not the geometry you're used to. Non-Euclidean geometry turns out to be fundamental to huge regions of math, including the math that describes the physical space we actually inhabit. (We'll come back to that in a few pages.) We *could* have refused to discover it on uptight Euclid-purist grounds. But it would be our loss.

Here's another place where a careful balance between formal logic and intuition is called for. Suppose a triangle is isosceles

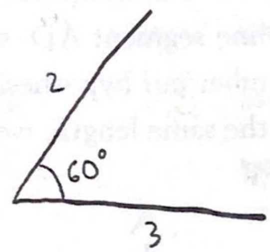


which is to say the sides AB* and AC are equal in length. Here's a theorem: the angles at B and C are equal as well.

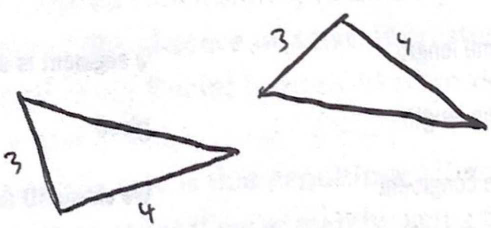
This statement is called the *pons asinorum,* the "bridge of asses," because it's something almost all of us have to be carefully led across. Euclid's proof has somewhat more to it than the business with the right angles above. We're a little in medias res here, since in a real geometry class we'd arrive at the ass-bridge only after several weeks of prep; so

* In geometry we like to refer to the line segment joining points named A and B simply as AB, like the Baltimore-Washington Parkway but without the Parkway.

let's take for granted Euclid's Proposition 4 of book I, which says that if you know two side lengths of a triangle and you know the angle between those two sides, then you know the remaining side length and the remaining two angles, too. That is, if I draw this:



there's only one way to "fill in" the rest of the triangle. Another way to say the same thing: if I have two different triangles that have two side lengths and the angle between them in common, then the two triangles have *all* their angles and *all* their side lengths in common; they are, as the geometer's lingo has it, "congruent."



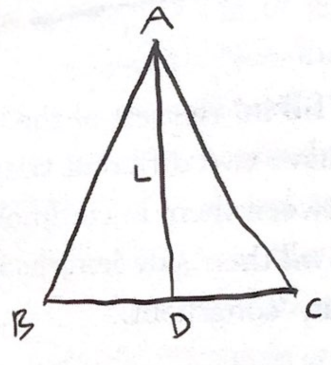
We invoked this fact already in the case where the angle between the two sides is a right angle, and I think the fact feels just as clear to the mind whatever the angle is.

(It's also true, by the way, that if the three side lengths of two triangles match up, the two triangles must be congruent; if the lengths are 3, 4, and 5, for instance, the triangle *must* be the right triangle I drew above. But this is less obvious, and Euclid proves it only a bit later, as Proposition I.8. If you think it *is* obvious, consider this: What about a four-sided figure? Remember the rhombus we just encountered; same four side lengths as a square, but definitely not a square.)

Now for the pons asinorum. Here's how a two-column proof might look.

Let L be a line through A which cuts angle BAC in half	**okay, I'll let you**
Let D be the point where L intersects BC	**still no objection**

Hey, me again, I know we're in midproof here, but we made a new point and invoked a new line segment AD, so we'd better update our picture! By the way, remember our hypothesis that our triangle is isosceles, so AB and AC have the same length; we're about to use that.



AD and AD have the same length	**a segment is equal to itself**
AB and AC have the same length	**given**
angles BAD and CAD are congruent	**We chose AD to cut angle BAC in half**
triangles ABD and ACD are congruent	**Euclid I.4, told you we'd need this**
angle B and angle C are equal	**corresponding angles in congruent triangles are equal**

QED.*

This proof has more to it than the first one we saw, because you actually have to *make* something; you made up a new line L and give the name D to the point where L hits BC. That allows you to identify B and C with edges of two newborn triangles ABD and ACD, which we then show are congruent.

* For "*Quod Erat Demonstrandum*," meaning "That which was to be proved," a little Latin bat-flip we like to execute at the end of our proofs for pizzazz. In my high school math crew we often substituted "AYD," meaning "And You're Done."

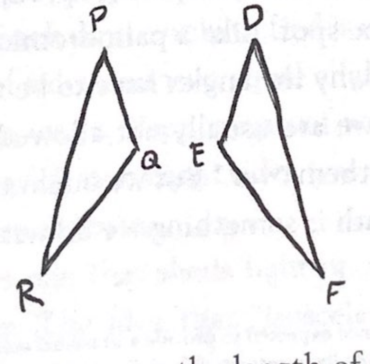
But there's a slicker way, written down about six hundred years after Euclid by Pappus of Alexandria, another North African geometer, in his compendium *Synagogue* (which in the ancient world could refer to a collection of geometrical propositions, not just a collection of Jews at prayer).

AB and AC have the same length	**given**
the angle at A equals the angle at A	**an angle is equal to itself**
AC and AB have the same length	**you already said that, what are you up to, Pappus?**
Triangles BAC and CAB are congruent	**Euclid I.4 again**
angle B and angle C are equal	**corresponding angles in congruent triangles are equal**

Wait, what happened? It seemed like we were doing nothing, and then all at once the desired conclusion appeared out of that nothing, like a rabbit jumping out of the absence of a hat. It creates a certain unease. It was not the sort of thing Euclid himself liked to do. But it is, by my lights at any rate, a true proof.

The key to Pappus's insight is that penultimate line: triangles BAC and CAB are congruent. It seems as if we're merely saying that a triangle is the same as itself, which looks like a triviality. But look more carefully.

What, really, are we saying when we say that two different triangles, PQR and DEF, are congruent?



We're saying six things in one: the length of PQ is the same as the length of DE, the length of PR is the length of DF, the length of QR is the length of EF, the angle at P is the same as the angle at D, the angle at Q is the angle at E, and the angle at R is the angle at F.

Is PQR congruent to DFE? Not in this picture, no, because PQ does *not* have the same length as the corresponding side DF.

If we take the definition of congruence seriously—and we're being geometers, so taking definitions seriously is kind of our thing—then DEF and DFE are not congruent to each other, *despite being the same triangle*. Because DE and DF don't have the same length.

But in the proof of the pons asinorum, we're saying that our isosceles triangle, when you think of it as triangle BAC, is the same as the triangle when thought of as CAB. That is *not* an empty statement. If I tell you the name "ANNA" is the same backward and forward, I'm really telling you something about the name: that it's a palindrome. To object to the very concept of a palindrome by saying, "Of course they're the same, it consists of two *A*'s and two *N*'s whichever order you write it in," would be pure perverseness.

In fact, "palindromic" would be a good name for a triangle like BAC, which is congruent to the triangle CAB you get when you write the vertices in the opposite order. And it was by thinking this way that Pappus was able to give his faster path across the pons, without having to invoke any extra lines or points at all.

And yet even Pappus's proof doesn't quite capture *why* an isosceles triangle has two equal angles. It does come closer. This notion that the isosceles triangle is a palindrome, that it stays the same when written backward, records something I'll bet your intuition also tells you—that the triangle is unchanged when you pick it up, flip it over, and lay it back down again in the same spot. Like a palindromic word, it has a *symmetry*. That, one feels, is why the angles have to be the same.

In geometry class we are usually not allowed to talk about picking up shapes and turning them over.* But we ought to be. As abstract as we may try to make it, math is something we do with our body. Geometry

* The Common Core standards, once expected to provide a universal scaffolding for K–12 math education in the United States, but now decidedly in retreat, did ask for the symmetry point of view to be covered in geometry class. One hopes some symmetry arguments will be left behind as the Common Core recedes, like glacial moraine.

most of all. Sometimes literally; every working mathematician has found themself drawing invisible figures with hand gestures, and at least one study has found that children asked to act out a geometric question with their body become more likely to arrive at the correct conclusion.* Poincaré himself was said to rely on his sense of motion when reasoning geometrically. He was not a visualizer, and his recollection for faces and figures was poor; when he needed to draw a picture from memory, he said, he remembered not what it looked like but how his eyes had moved along it.

EQUAL ARMS

What does the word "isosceles" really mean? Well, it means two sides of the triangle are equal. Literally, in Greek, it refers to the two σκέλη (*skeli*), or "legs." In Chinese, 等腰 means "equal waists"; in Hebrew an isosceles triangle is one with "equal calves," in Russian "equal arms." In every case, we seem to agree that what it means to be isosceles is to have two sides equal. But why? Why not define an isosceles triangle to be one that has two angles equal? You can probably see (and indeed the whole point of the pons asinorum is to prove!) that two sides being equal means two angles are equal, and vice versa. In other words, the two definitions are equivalent; they pick out the same collection of triangles. But I wouldn't say they're the *same* definition.

Nor are they the only option. It would be more modern in flavor to define an isosceles triangle as a palindromic one: a triangle you can pick up, flip over, and place back down, only to find it unchanged. That such a triangle has two equal sides and two equal angles is just about automatic. In this geometric world, Pappus's proof would be the means of showing that a triangle with two equal sides was isosceles; that the triangles BAC and CAB are the same.

A good definition is one that sheds light on situations beyond the ones it was devised for. The idea that "isosceles" means "unchanged

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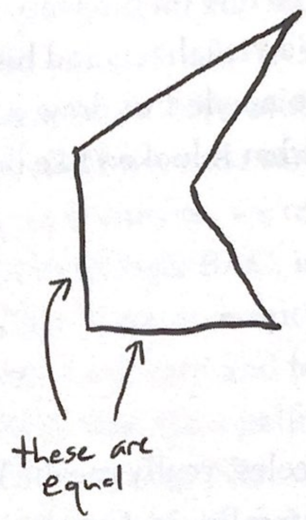
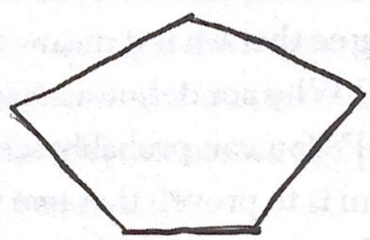
* Though no more likely to be able to construct a formal proof of that conclusion!

when flipped over" gives us a good idea of what we should mean by an isosceles trapezoid, or an isosceles pentagon. You *could* say that an isosceles pentagon is one that has two sides equal; then you'll be admitting saggy, lopsided pentagons like this one into the fold:



But do you want to? Surely a pentagon like this handsome figure



is more what one means by isosceles. Indeed, in your schoolbook, an "isosceles trapezoid" isn't one with two equal sides, or with two equal angles; it is one that can be flipped without changing it. The post-Euclidean notion of symmetry has crept in, and it's there because our minds are built to find it. More and more geometry classes are placing the idea of symmetry at the center, and building structures of proof starting from there. It's not Euclid, but it's where geometry is now.