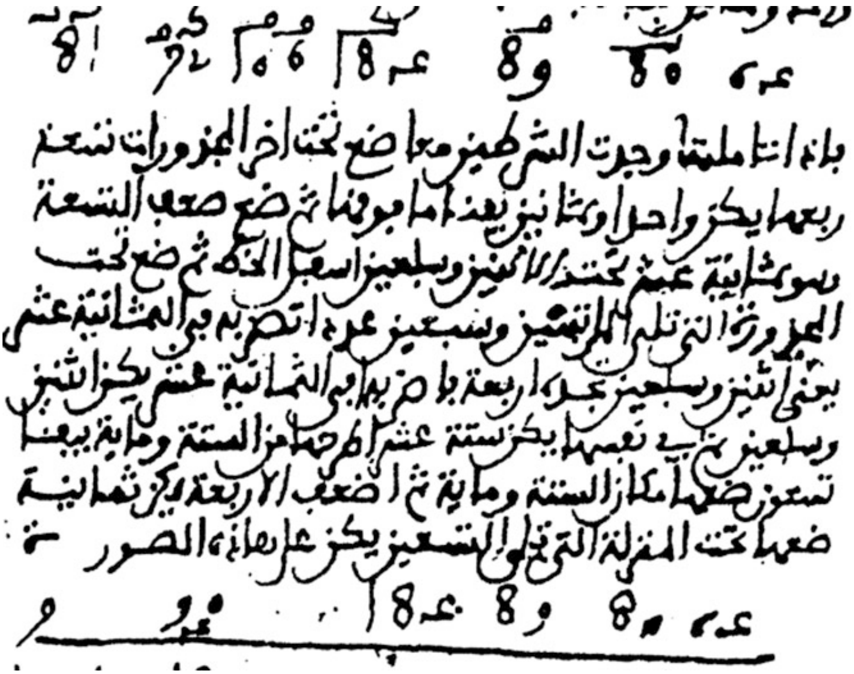
The symbol for equality, the letter *lām* (ل), appears in both equations above. In the second equation the second symbol from the right is the Arabic word *lā*, which, in an algebraic context, functions as a minus.

The image below shows two equations, of degrees six and nine respectively, from the work of the fifteenth-century writer Aḥmad al-Qatrawānī, who was born in Upper Egypt but taught in Tunis. It is the first example we know of an Arabic writer extracting roots of polynomials using algebraic symbolism rather than the tables we saw in al-Samaw'al's work.



The reader may use the information we have given about algebraic symbolism to read the top polynomial (left to right) as

$$81x^6 + 72x^5 + 106x^4 + 184x^3 + 89x^2 + 80x + 64.$$

In the case of this polynomial, the text shows how to extract its square root.

9 ʿUmar al-Khayyāmī and the Cubic Equation

9.1 The Background to ʿUmar's Work

An older contemporary of al-Samaw'al was ʿUmar al-Khayyāmī, whose treatment of cubic equations is found in his book *Algebra*. He completed this work in

Samarqand and dedicated it to the chief judge of that city, Abū Ṭāhir, and in the preface to this work he refers to his harried existence up to then

> I have always desired to investigate all types of theorems …, giving proofs for my distinctions, because I know how urgently this is needed in the solution of difficult problems. However, I have not been able to find time to complete this work, or to concentrate my thoughts on it, hindered as I have been by troublesome obstacles.

When ʿUmar did have the security to concentrate on a problem his powers of intellect were remarkable. One of his biographers, al-Bayhaqī, tells how ʿUmar read a book seven times in Isfahan and memorized it. When he returned he wrote it out from memory, and subsequent comparison with the original revealed very few discrepancies. However, ʿUmar's feats of intellect were by no means confined to a remarkable memory, as we shall see in the sections from his great work, *Algebra*.

In his introduction to this work ʿUmar mentions that no algebraic treatment of the problems he is going to discuss has come down from the ancients, but that among the modern writers Abū ʿAbdallāh al-Māhānī wrote an algebraic analysis of a lemma Archimedes used in the problem from his work *Sphere and Cylinder* II,4 which we mentioned earlier, the problem of cutting a sphere by a plane so that the volumes of the two segments of the sphere are to one another in a given ratio. Archimedes showed that this problem can be solved if a line segment a can be divided into two parts b and c so that c is to a given length as a given area is to b^2. If we let $b = x$, so $c = a - x$, the proportion may be written as $x^3 + m = nx^2$, where m is the product of the given length and area. Khayyam tells us that neither al-Māhānī, who lived from 825–888 (and would therefore have been contemporary with al-Khwārizmī), nor Thābit could solve this equation, but a mathematician of the next generation, Abū Jaʿfar al-Khāzin, did solve it by means of intersecting conic sections. Then, following Abū Jaʿfar, various mathematicians tried to solve special kinds of these equations involving cubes, but no one had tried to enumerate all possible equations of this type and solve them all. This ʿUmar says, he will do in this treatise.

9.2 ʿUmar's Classification of Cubic Equations

We now give an account of some parts of his treatise *Algebra*, and we emphasize that although we shall speak of "equations" and "coefficients" ʿUmar did not write these symbolically, for he used only words, even for the numbers.

In the first part of his treatise ʿUmar lists all types of equations in which no term of degree higher than three occurs. In ʿUmar's equations all terms appear with positive coefficients so that, whereas we would see $x^3–3x + 8 = 0$ and $x^3 + 3x - 8 = 0$ as being of the same type, ʿUmar viewed them as being different. He would have expressed the first as "Cube and numbers equal sides" ($x^3 + 8 = 3x$) and would have seen it as distinct from "Cube and sides equal numbers" ($x^3 + 3x = 8$). Thus, he arrives at 25 species of equations, and, in the remainder of the treatise, he shows how these may be solved—11 by Euclidean methods and 14 by conic sections. For each of these 14 species ʿUmar gives a short section showing how the conics can be used to produce a line segment from which solids that satisfy the required relation can be constructed. The student who reads the English translation by D. S. Kasir will find ʿUmar's arguments quite clear and will enjoy, as well, the many interesting asides on the history of various types of equations.

9.3 ʿUmar's Treatment of $x^3 + mx = n$

Preliminaries

The equation we are going to discuss is "cube and sides equal a number," that is, "$x^3 + mx = n$" and, to understand the section that we have chosen to present from this work, the reader must recall that if ABC is a parabola with vertex B and parameter p and if x is any abscissa and $\mathbf{y}$ the corresponding ordinate then $y^2 = p \cdot x$.

ʿUmar begins with a lemma about solid figures called parallelopipeds, solids with three pairs of parallel faces (Fig. 8a). When all the faces are rectangles (Fig. 8b), as in the case of a brick, the solid is called a rectangular parallelopiped. One face of a parallelopiped is arbitrarily designated as its base, and ʿUmar's lemma concerns the case in which the base is a square.

Lemma. Given a rectangular parallelopiped ABGDE (Fig. 9), whose base is the square ABGD = a^2 and whose height is c, and given another square MH = b^2, construct on MH a rectangular parallelopiped equal to the given solid.

Solution. Use Euclidean geometry to construct a line segment k so that $a:b = b:k$; and then construct h so that $a:k = h:c$. Then the solid whose base is b^2 and whose height is h is equal to the given solid.

Proof. $a:b$ = b:k implies $a^2:b^2 = (a:b) \cdot (a:b) = (a:b) \cdot (b:k) = a:k$, but $a:k = h:c$, and so $a^2: b^2 = h:c$, and this implies $a^2 \cdot c = b^2 \cdot h$. Thus the solid whose base is b^2 and whose height is h is equal to the given solid $a^2 \cdot$ c, and that is what we wanted to show.

Algebraically this lemma asks for the root of $a^2 \cdot c = b^2 \cdot x$, given a, c and b, and Khayyām's constructions obtain this solution ($a^2 \cdot c/b^2$) by first obtaining $k = b^2/a$ and then $h = (ac)/k = a^2 \cdot c/b^2$. The principal fact, taken as known by ʿUmar, is that given any three straight-line segments a, b, c it is possible to find a fourth segment

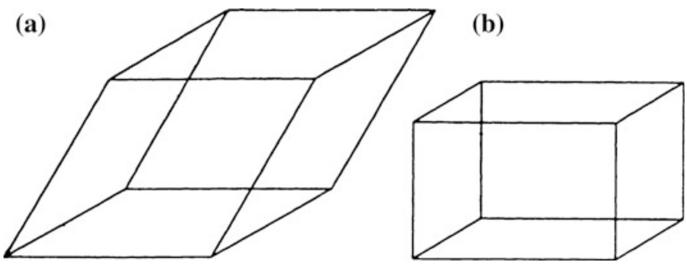


Fig. 8

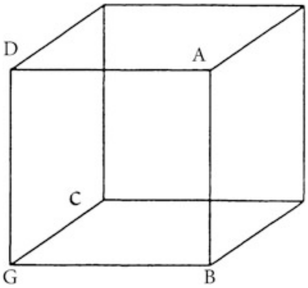


Fig. 9

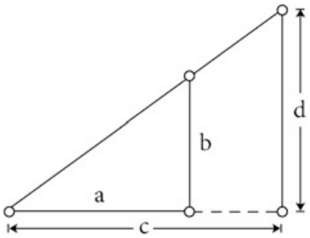


Fig. 10

d so that $a{:}b = c.d.$ The segment d is called "the fourth proportional", and Fig. 10 shows how d can be constructed.

9.4 The Main Discussion

ʿUmar now comes to his first nontrivial equation, which he describes as "Cube and sides equal a number," i.e., the case we would write as $x^3 + mx = n$, where m and n are positive. For this he gives the following procedure: Let b be the side of a square that is equal to the number of roots, i.e., $b^2 = m$, and let h be the height of the rectangular parallelopiped whose base is b^2 and whose volume is n. (The construction of h follows immediately from the previous lemma.) Now take a parabola (Fig. 11) whose vertex is B, axis BZ and parameter b, and place h perpendicular to BZ at B. On h as diameter describe a semicircle and let it cut the parabola at D. From D drop DE perpendicular to h and the ordinate DZ perpendicular to BZ. Then DZ = EB and with y = BE it follows that $y^3 + my = n$.

Proof. Let BZ = x. By the properties of the parabola $y^2 = bx$ and, by the properties of the circle, $x^2 = y(h - y)$. But the first equality may be written as $x{:}y = y{:}b$ and the second as $x{:}y = (h - y){:}x$. Thus $(h - y){:}x = x{:}y = y{:}b$, or, inverting, $b{:}y = y{:}x = x{:}(h - y)$. Hence $b^2{:}y^2 = (b{:}y)(b{:}y) = (y{:}x)(x{:}(h - y)) = y{:}(h - y)$, and thus $b^2 \cdot (h - y) = y \cdot y^2$, i.e. $b^2 \cdot h - b^2 \cdot y = y^3$. Therefore, if we add $b^2 \cdot y$ to both sides it follows that $b^2 \cdot h = y^3 + b^2 \cdot y$. If we then substitute for $b^2 \cdot h$ its equal, n,

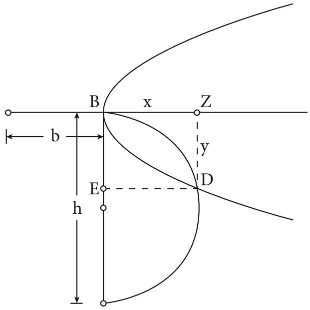


Fig. 11

and for b^2 its equal, m, we may conclude that $y^3 + my = n$, which was the equation we wanted to solve.

We can write the argument a bit more briefly as $y^2 = bx$ implies $y^4 = b^2x^2$, and $x^2 = y(h - y)$ implies $b^2x^2 = b^2y(h - y)$. Thus, $y^4 = b^2x^2 = b^2y(h - y)$ and so, because $y \neq 0$,

$$y^3 = b^2(h - y), \quad \text{i.e.} \quad y^3 + my = y^3 + b^2 \cdot y = b^2 \cdot h = n$$

and the equation is solved.

9.5 ʿUmar's Discussion of the Number of Roots

Throughout the discussion ʿUmar is careful to warn the reader that a particular case may have more than one solution (or, as we should say, more than one positive real root) or that it may have no solutions. What happens in any given case depends on whether the conic sections he is using intersect in none, in one or in two points. For example, he obtains the solution to $x^3 + n = mx$ by intersecting a parabola and hyperbola and notices that the two curves may not intersect, in which case there would be no solution, but if they do then they either intersect tangentially or at two points. In our modern terminology, we would express this by saying the equation $x^3 + n = mx$ either has no positive real solution or two of them. In the latter case, the two could be a single root, a repeated root, corresponding to a factor $(x - a)^2$, or two different roots. Again, in the case of $x^3 + n = mx^2$ he notes that if $\sqrt[3]{n} \geq m$ then there is no solution. For, if $\sqrt[3]{n} \geq m$ then

$$n = \left(\sqrt[3]{n}\right)^3 = \sqrt[3]{n}\left(\sqrt[3]{n}\right)^2 \geq m\left(\sqrt[3]{n}\right)^2. \tag{1}$$

This implies that if x is any solution then $x > \sqrt[3]{n}$ for

$$x^3 + n = mx^2 \quad \text{implies} \quad mx^2 > n. \tag{2}$$