Chapter 7
Number Theory and Combinatorics in the Islamic World

1 Number Theory

Number theory has a rich ancient tradition, much of it being found in Books VII–IX of Euclid's *Elements*. Among the beautiful results in these three books, one finds a proof that there are infinitely many prime numbers, and that if $2^n - 1$ is a prime then $2^{n-1}(2^n - 1)$ is a *perfect* number, i.e., is equal to the sum of its proper divisors.[1] Moreover, in Book X, one finds a rule for generating squares of whole numbers whose sum is also a square.

Not too long after Euclid, Eratosthenes developed his famous sieve for finding the primes in a sequence of the first n integers, and, some centuries later, Diophantos, among other investigations, solved the problem of finding two rational numbers such that when either is added to the square of the other the result is the square of a rational number. And in the same work, the *Arithmetica*,[2] he sets forth an algebraic method of finding rational solutions for indeterminate equations, i.e., equations such as $x^2 + y^2 = z^2$ that have more than one solution. Both the *Arithmetica* and the *Elements* were well known in medieval Islam.[3]

At some point between the time of Euclid and Diophantos (probably around 100 A.D), Nicomachos of Gerasa wrote his *Introduction to Arithmetic* in which he discussed, among other topics, figured numbers (see below) and gave the fourth perfect number, 8128.[4]

One of the earliest results concerning the number theory that was proved in medieval Islam was that of Thābit b. Qurra who translated both Euclid and Nicomachos and proved a sufficient condition for two whole numbers to be

[1]The first two perfect numbers are 6 and 28.

[2]At that time in the Greek world, "arithmetic" (*arithmētikē*) referred to what we call "the theory of numbers." Our "arithmetic" the Greeks called "logistic" (*logistikē*).

[3]However, of the 13 books of Diophantus's *Arithmetica* only six books survive in Greek and another four those only in Arabic.

[4]The earliest mention of the fifth perfect number, in the 15th century, is 33,550,336.

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amicable. This notion, which seems to be an extension of the idea of perfect number, went back to the ancient Greeks who called two numbers amicable if each is equal to the sum of the proper divisors of the other. (The numbers 220 and 284 were standard ancient examples of such numbers.) Thābit's theorem states that: If p, q, and r are primes such that $p = 3 \times 2^{n-1} - 1$, $q = 3 \times 2^n - 1$, and $r = 9 \times 2^{2n-1} - 1$ then $2^n pq$ and $2^n r$ are amicable numbers.

1.1 Representing Rational Numbers as Sums of Squares

We mentioned, above, that Diophantos gave a method for finding rational solutions, x, y, and z, to the equation $x^2 + y^2 = z^2$. In the thirteenth-century, in II, 1 of his *Algebra,* Ibn al-Bannā' deals with the problem of finding rational solutions to the equation $x^2 + y^2 = z^2$. This number theoretic material occurs in the context of finding solutions to problems of three types: (1) dividing 10 into two parts satisfying certain conditions, (2) dividing money among a certain number of men so that certain conditions are satisfied, and (3) a certain sum of money is increased and decreased according to certain conditions. But, to prepare his reader to solve such problems he states the following rules:

1. If a and b are any two numbers[5] such that $a/b = 3/4$ then $a^2 + b^2$ is a square of a rational number.
2. If a is a square it may be expressed as the sum of two squares. His abbreviated demonstration of this is: "This is because there exist two squares whose square is a square. The given square is then decomposed according to their ratio."
3. If a is not a square, then *if* there exist *whole* numbers x and y satisfying $a = x^2 + y^2$ there also exists another, different, pair of numbers, w and z, such that $a = w^2 + z^2$.
4. He concludes with the following procedure for deciding whether a whole number can be expressed as the sum of two squares:

> You may know whether it has two square parts by subtracting from it the first of the natural squares, i.e. 'one.' And if the difference has a [whole number] root [then you have expressed it as the sum of two squares. But if not, one subtracts the second square, which is 'four' and one examines the remainder And one proceeds step-by-step in this fashion.
>
> If it is one of those numbers that cannot be expressed as the sum of two squares this will become evident with whole numbers. For if it cannot be decomposed into two whole number squares neither can it be decomposed into squares of fractions. Keep this in mind.

Only in Rule 3 does he refer specifically to whole numbers, so one assumes that when not further specified "number" refers to rational numbers in general.

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[5]Although this is true for any two rational numbers Ibn al-Bannā' simply says "any two numbers."