Gauss sped through the curriculum at the Collegium and enrolled at the University of Göttingen, sixty miles from Brunswick, rather than the Duchy's official university in nearby Helmstedt, most likely due to Göttingen's superior mathematics library. Surprisingly, Göttingen's records show that Gauss borrowed far more books in the humanities than in mathematics. Gauss's notebooks tell us that he had a much higher regard for his classics professor than for his mathematics professor. However, mathematical accomplishment quickly won out.

The Greeks had known that regular polygons with 3 or 5 or 15 sides could be constructed with a straight edge and compass. So could any regular polygon having a number of sides that was a power of 2 times 3, 5, or 15, and that is where the boundary of the field stood for two millennia until 1796 when Gauss discovered that a seventeen-sided regular polygon could also be constructed with the classical geometrical tools: the straight edge and compass. He quickly generalized his result to any regular polygon with a number of sides that is a product of a power of 2 and any number of Fermat primes. A Fermat prime is a prime number of the form $2^N + 1$, where N is itself a power of 2. Fermat (1601–1665) thought that all numbers of the form $2^N + 1$, where N is a power of 2, are prime numbers, and it is easy to see that 3, 5, 17, and 257 have this property. With only a little bit of brute force you can demonstrate, as Fermat must have done, that 65,537 ($= 2^{16} + 1$) is a prime. The next candidate Fermat prime is 4,294,967,297 ($= 2^{32} + 1$). We cannot fault Fermat for not finding 641 as its smallest prime divisor. It took a century for the great mathematician Leonhard Euler to find that. In his notebooks, Gauss speculated that there are no other Fermat primes. To date, none have been found to exist.

Gauss was so overjoyed at this result that it convinced him to pursue a career in mathematics. After two years at Göttingen, he realized that no one on the faculty could really be of any assistance to him, so he went home to Brunswick to write his doctoral dissertation. For his topic he chose the fundamental theorem of algebra, that every polynomial equation of degree n with complex coefficients has exactly n roots in the complex numbers. His dissertation was the first of what would be four proofs throughout his career.

Freed of the need to write a set piece, Gauss turned his attention to number theory. Number theory goes back to the Greeks with Euclid's proofs of the infinity of prime numbers and the form of even perfect numbers being two of the earliest results in the field. From time to time, new results were added or new conjectures made. In the seventeenth century, the French mathematician Pierre de Fermat, a contemporary of Descartes, made his famous conjecture that the equation $x^n + y^n = z^n$ has no nontrivial solutions in integers for $n > 2$. The Arabs made same progress for the cases $n = 3$ and $n = 4$, however, the complete proof of Fermat's Last Theorem did not come until 1995. In the fifty years before Gauss, Lagrange had provided the first proof that every integer can be expressed as the sum of no more than four squares and Goldbach had conjectured that every even number other than 2 can be expressed as the sum of two primes. In 1770, the English mathematician published the statement of a theorem that had first been proposed by his former student John Wilson.

The integer p *is a prime if and only if* p *evenly divides* $(p - 1)! + 1$.

Recall that $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n - 1) \cdot n$. Given the speed with which $n!$ (n factorial) grows, Waring despaired of ever having a notation that would enable a proof of the conjecture whose truth could barely be established in particular cases.

When Gauss began work on his epochal *Disquisitiones arithmeticae* (*Arithmetic disquisitions*) excerpted here, number theory was merely a collection of isolated results. Upon hearing of Waring's despair, Gauss is supposed to have remarked that mathematics is concerned with notions rather than with notations. In the *Disquisitiones*, he introduced the notion of congruence and in so doing unified number theory. Two integers x and y are said to be *congruent* modulo the integer z if and only if $(x - y)$ is evenly divisible by z. Following Gauss, we express this congruence relationship as

$$x \equiv y \ (mod\ z)$$

This powerful analytical method was the foundation of the *Disquisitiones arithmeticae*, a compendium of Gauss's result in number theory, which he published at the age of twenty-four. It is divided into seven sections:

1. congruences in general
2. congruences of the first degree
3. residues of powers
4. congruences of the second degree
5. quadratic forms
6. applications
7. division of the circle

With the notion of congruence and Gauss's compact notation for it, he could easily state Wilson's theorem as

The integer p *is a prime if and only if* $(p - 1)! \equiv -1 \ (mod\ p)$

and he could also state and prove the complement to Wilson's theorem:

If n *is a composite number other than 4, then* $(n - 1)! \equiv 0 \ (mod\ n)$.

These results only hinted at the power of congruence as a mathematical tool. In 1795, while still at the Collegium, Gauss had discovered the law of quadratic reciprocity, which had only been stated and incompletely proven a decade earlier by the thirty-three-year-old French mathematician Adrien-Marie Legendre. In the language of congruences, the law of quadratic reciprocity states:

If p and q are primes not both congruent to 3 modulo 4 then either both $x^2 \equiv p \pmod{q}$ and $x^2 \equiv q \pmod{p}$ are solvable

or

neither $x^2 \equiv p \pmod{q}$ nor $x^2 \equiv q \pmod{p}$ is solvable.

If p and q are primes both congruent to 3 modulo 4 then exactly one of the equations

$$x^2 \equiv p \pmod{q} \text{ is } x^2 \equiv q \pmod{p}.$$

Notice that this theorem comes nowhere close to resolving the question of whether p is in fact a quadratic residue modulo q *in isolation*! It merely allows one difficult numeric calculation, such as whether 257 is a prime modulo 65,537, to be replaced by the easier numeric calculation of whether 65,537 is a prime modulo 257. (The first step is to reduce 65,537 to its residue modulo 257, which is 2, and then determine whether 2 is a quadratic residue modulo 257.)

Gauss considered the law of quadratic reciprocity to be the *theorem aureum*— the golden theorem — or the *gemma arithmeticae* — the gem of arithmetic. He considered arithmetic itself, as he called number theory, to be the *queen of mathematics*, which he called the *queen of sciences*.

Early in 1801, the Duke of Brunswick raised Gauss's stipend. However, Gauss felt as though he had done little to merit the raise. With the *Disquisitiones arithmeticae* in press, Gauss sought a new challenge and turned his attention to planetary theory. In January 1801, the Italian astronomer Joseph Piazzi had briefly observed what he thought to be a new planet before losing track of it. Gauss spent much of 1801 improving the theory of planetary perturbation to utilize true ellipses rather than circular approximations. Gauss predicted the mysterious body was an asteroid rather than a new planet. At the end of the year, astronomers found the asteroid Ceres exactly where Gauss's improved method said it should be. The discovery of Ceres earned Gauss genuine international fame. In January 1802, the Academy of Sciences in the Russian capital of St Petersburg elected him a corresponding member. Gauss felt as though he had indeed merited the increase in the Duke's stipend.

The Duke raised Gauss's stipend again in 1803. The increased income may have prompted Gauss to think about his personal situation. In 1805, Gauss surprised everyone around him by announcing his engagement to Johanna Osthoff after a year-long courtship. He wrote his friend Bolyai, "For three days now this angel, almost too heavenly for our earth, has been my fiancée . . . Life lies before me like an eternal spring with radiant colors." They married on October 9, 1805. For a brief time, Gauss did seem to be in the spring of his life. His patron rewarded him with a hefty increase following his marriage, perhaps motivated by an offer of a position in St. Petersburg. The Gauss's first child Joseph, named for the discoverer of Ceres, was born on August 21, 1806. A daughter christened Wilhelmina followed a year and a half later on February 29, 1808.

Unfortunately for Gauss, his blissful spring would not last long. In November 1806, Duke Ferdinand died from a wound he suffered losing the Battle of Auerstädt to Napoleon the previous month. Following Napoleon's victories in Germany, Göttingen found itself in the Kingdom of Westphalia, a French vassal state. As a professor, Gauss had to pay 2,000 francs tax, a small fortune in those days. The astronomer Olbers sent Gauss the necessary funds to pay the tax; however, Gauss refused his benevolence. Then Gauss received a letter from the French mathematician Laplace who said he considered it an honor to pay the tax and lift the obligation from Gauss. Once again Gauss declined the offer, but not out of any animosity for the Frenchman. Gauss had a great respect for Laplace and Laplace for Gauss. When Laplace was once asked who was the greatest mathematician in Germany, he immediately replied with the name of Pfaff, who had nominally supervised Gauss's doctoral dissertation. When asked why he hadn't named Gauss, Laplace immediately replied, "Gauss is the greatest mathematician in the world." In the end, an anonymous donor sent Gauss the money to pay the tax. Unable to repay the donor, Gauss made regular donations to charity with interest as if paying of a loan for the amount donated.

Gauss would not have a long marriage to Johanna. She died a month after giving birth to their third child, Ludwig, in 1809. Sadly, Ludwig died five months later. Within a year of Johanna's death, Gauss married Minna Waldeck, Johanna's best friend. We can only guess that Gauss may have been motivated by a need to find a stepmother for his three children. Gauss had three more children by Minna before she became ill, first with tuberculosis and then with what was diagnosed as hysterical neurosis. Gauss and Minna were perpetually unhappy until her death in 1831.

At the beginning of the nineteenth century, Paris remained the center of the mathematical world. Göttingen was at best a remote outpost. Gauss had occasional correspondence with the mathematical giants in France, but he never troubled himself to visit Paris. There were no mathematicians in Germany who came close to him in stature during the prime of his career and few worthy of correspondence. Given the mediocre state of affairs in the German mathematics community, it is not surprising that Gauss chose to serve as Professor of Astronomy and Director of the Observatory at Göttingen. Had he been Professor of Mathematics, he would have been required to spend his time teaching Mathematics to indifferent undergraduates.

Not having mathematics classes to teach may have given Gauss the free time to pursue the consequences of denying Euclid's parallel postulate, research he had begun in his student days and then put aside. He seems to have done this work almost against his will, hardly bearing the thought that the parallel postulate might not be true. He was never able to allow himself to publish this work. We know it only through his notebooks.

Gauss's contemporaries considered him to be a mathematical scientist with strong interests in applied and even empirical mathematics. His interest in geodesy, the mathematics of surveying and representing the land, is a good example of his strong empirical leanings. He initially worked on surveying problems in his early twenties, then set that interest aside for nearly two decades. In 1817, at the age of forty he returned to this subject, taking responsibility for a survey of the state of Hanover. For several years Gauss spent his summers surveying the land and spending much of the rest of the year analyzing the data. Dissatisfied with the standard geodetic measurement techniques based on sighting lamps or flares, Gauss invented a new method using a device called a *heliotrope*. It employed mirrors to deflect light rays to small aperture telescopes.

Gauss employed his geodetic work to find empirical support for non-Euclidean geometry. As part of one of his surveys he measured the angles of the triangle formed by the mountain tops of the northern German peaks Hohenhagen, Brocken, and Inselsberg. The measurement of this triangle, with sides between 45 and 70 miles long, proved inconclusive. Gauss calculated the sum of the angles of the triangle to be 180° 0′ 15″, 1 part in 43,200, close enough to 180°, to be due to measuring error. Thanks to Einstein's theory of general relativity, we now know that the sum of the angles of such a triangle exceeds 180° by 10^{-17}″, 1 part in 10^{21}!

Without any empirical data, Gauss chose to not to publish any of his work on non-Euclidean geometry. However, when Gauss received word in 1831 that the Hungarian mathematician János Bolyai had published work demonstrating the consistency of a non-Euclidean geometry, he responded to János's father Farkas, an old friend:

> *To praise it would amount to praising myself. For the entire content of the work . . . coincides almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years.*

Gauss thought that he was merely reporting a fact. Both Bolyais took it as a great offense and as an attempt to steal priority.

In 1824, Gauss received a substantial salary increase, his first raise since 1807. A year later, he received a large bonus for the surveying work. This newfound financial bounty came at a very fortunate point in his life, just as Gauss was beginning to suffer from asthma and a heart ailment. By 1825, the physical burden of the summer surveys became too much for him to bear and he satisfied himself with supervising the surveys and doing all of the calculations. It has been estimated that he handled more than a million pieces of numeric data by himself. Given his computational talents, it is hardly surprising that even late in his life Gauss would find new fields for his talents. In the 1840s, he took on the task of putting the university's pension fund on a sound actuarial basis. He must have been especially good at investments. When he died, his estate was equal to two hundred times his annual income.

Over the years, Gauss began to attract a handful of students to his occasional mathematical lectures. Anyone who attended his lectures on number theory in 1809, the theory of curved surfaces in 1827, or the method of least squares in 1851 should have considered himself exceptionally fortunate. Bernhard Riemann and Richard Dedekind, each included in this book, were among the fortunate. Their generation established Göttingen as the center of the mathematical universe.

Gauss resisted his physical ailments until he was well into his seventies. He finally died on February 23, 1855, two months short of his seventy-eighth birthday.

In his will, Gauss stipulated that a seventeen-sided regular polygon be carved into his gravestone. However, that was not to be. The mason charged with the task thought that viewers would confuse a seventeen-sided regular polygon with a circle, so he carved a seventeen-pointed star. Although the mason may not have followed Gauss's instructions, he did mark Gauss as a star, the greatest star in the mathematical firmament.