MA 330 In-Class Work

Proof of the infinitude of primes, essentially due to Euler, though he missed some details. This is taken from *Proofs from the Book*, by M. Aigner and G. Ziegler, Chapter 1.

For any real number x, let $\pi(x) := \#\{p \le x : p \text{ prime}\}$, i.e. $\pi(x)$ is the number of primes less than or equal to x. Number the primes $\mathbb{P} := \{p_1, p_2, p_3, \ldots\}$ in increasing order. Recall that the natural log function is defined by

$$\ln(x) = \int_1^x \frac{1}{t}\, dt\,.$$

The main idea is to compare the graph of $1/t$ with an "upper step function," namely the function that has the value $1/n$ for all x between n and $n+1$. Note that the area under the graph of the step function between 1 and n is greater than the area under the graph of $1/t$ between 1 and n.

So, because the natural log is defined as the area under the graph of $1/t$, for all x such that $n \le x \le n+1$ we see that

$$\ln(x) \le 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n} \le 1 + \sum_{m \in M(x)} \frac{1}{m}$$

where $M(x)$ is the set of all positive integers having prime divisors no larger than x.

The key step in the proof comes next! How exciting! What we need to do is show that we have the following equality:

$$1 + \sum_{m \in M(x)} \frac{1}{m} = \prod_{\substack{p \in \mathbb{P} \\ p \le x}} \left(\sum_{k \ge 0} \frac{1}{p^k} \right).$$

(Make sure you do this! It feels awesome to understand how this works!)

Once you have proved this claim, then note that the infinite series in parentheses on the right-hand side above is a geometric series, so

$$\sum_{k \ge 0} \frac{1}{p^k} = \frac{1}{1 - \frac{1}{p}}\,.$$

Thus, putting all this together, we have

$$\ln(x) \le \prod_{\substack{p \in \mathbb{P} \\ p \le x}} \frac{1}{1 - \frac{1}{p}} = \prod_{\substack{p \in \mathbb{P} \\ p \le x}} \frac{p}{p-1} = \prod_{k=1}^{\pi(x)} \frac{p_k}{p_k - 1}\,.$$

Since $p_k \ge k+1$ (this is "clear" if you think about it in the right way, since p_k is the k-th prime…remember, we ordered them p_1 then p_2 etc), we have

$$\frac{p_k}{p_k - 1} = 1 + \frac{1}{p_k - 1} \le 1 + \frac{1}{k} = \frac{k+1}{k}\,.$$

Therefore,

$$\ln(x) \le \prod_{k=1}^{\pi(x)} \frac{k+1}{k} = \pi(x) + 1\,.$$

(Why is that last equality true? Work through a few examples.)

We all know that $\ln(x)$ gets arbitrarily large as $x \to \infty$. So, since $\pi(x)$ is greater than $\ln(x)$, it must be that $\pi(x)$ gets arbitrarily large as $x \to \infty$ as well. Hence, there must be infinitely many primes.