became the father of Greek medicine and originator of the physicians' Hippocratic oath.)

Of the mathematical Hippocrates, we have scant biographical information. Aristotle wrote that, while a talented geometer, he "... seems in other respects to have been stupid and lacking in sense." This is an early example of the stereotype of the mathematician as being somewhat overwhelmed by the demands of everyday life. Legend has it that Hippocrates earned this reputation after being defrauded of his fortune by pirates, who apparently took him for an easy mark. Needing to make a financial recovery, he traveled to Athens and began teaching, thus becoming him one of the few individuals ever to enter the teaching profession for its financial rewards.

In any case, Hippocrates is remembered for two signal contributions to geometry. One was his composition of the first *Elements*, that is, the first exposition developing the theorems of geometry precisely and logically from a few given axioms or postulates. At least, he is credited with such a work, for nothing remains of it today. Whatever merits his book had were to be eclipsed, over a century later, by the brilliant *Elements* of Euclid, which essentially rendered Hippocrates' writings obsolete. Still, there is reason to believe that Euclid borrowed from his predecessor, and thus we owe much to Hippocrates for his great, if lost, treatise.

The other significant Hippocratean contribution—his quadrature of the lune—fortunately has survived, although admittedly its survival is tenuous and indirect. We do not have Hippocrates' own work, but Eudemus' account of it from around 335 B.C., and even here the situation is murky, because we do not really have Eudemus' account either. Rather, we have a summary by Simplicius from A.D. 530 that discussed the writings of Eudemus, who, in turn, had summarized the work of Hippocrates. The fact that the span between Simplicius and Hippocrates is almost a thousand years—roughly the time between us and Leif Erikson—indicates the immense difficulty historians face when considering the mathematics of the ancients. Nonetheless, there is no reason to doubt the general authenticity of the work in question.

Some Remarks on Quadrature

Before examining Hippocrates' lunes, we need to address the notion of "quadrature." It is obvious that the ancient Greeks were enthralled by the symmetries, the visual beauty, and the subtle logical structure of geometry. Particularly intriguing was the manner in which the simple and elementary could serve as foundation for the complex and intricate. This will become quite apparent in the next chapter as we follow Euclid through the development of some very sophisticated geometric propositions beginning with just a few basic axioms and postulates.

This enchantment with building the complex from the simple was also evident in the Greeks' geometric constructions. For them, the rules of the game required that all constructions be done only with compass and (unmarked) straightedge. These two fairly unsophisticated tools—allowing the geometer to produce the most perfect, uniform one-dimensional figure (the straight line) and the most perfect, uniform two-dimensional figure (the circle)—must have appealed to the Greek sensibilities for order, simplicity, and beauty. Moreover, these constructions were within reach of the technology of the day in a way that, for instance, constructing a parabola was not. Perhaps it is accurate to suggest that the aesthetic appeal of the straight line and circle reinforced the central position of straightedge and compass as geometric tools while, conversely and simultaneously, the physical availability of these tools enhanced the role to be played by straight lines and circles in the geometry of the Greeks.

The ancient mathematicians were consequently committed to, and limited by, the output of these tools. As we shall see, even the seemingly unsophisticated compass and straightedge can produce, in the hands of ingenious geometers, a rich and varied set of constructions, from the bisection of lines and angles, to the drawing of parallels and perpendiculars, to the creation of regular polygons of great beauty. But a considerably more challenging problem in the fifth century B.C. was that of the quadrature or squaring of a plane figure. To be precise:

□ The *quadrature* (or squaring) of a plane figure is the construction—using only compass and straightedge—of a square having area equal to that of the original plane figure. If the quadrature of a plane figure can be accomplished, we say that the figure is ***quadrable*** (or squarable).

That the quadrature problem appealed to the Greeks should come as no surprise. From a purely practical viewpoint, the determination of the area of an irregularly shaped figure is, of course, no easy matter. If such a figure could be replaced by an equivalent square, then determining the original area would have been reduced to the trivial matter of finding the area of that square.

Undoubtedly the Greeks' fascination with quadrature went far beyond the practical. For, if successfully accomplished, quadrature would impose the symmetric regularity of the square onto the asymmetric irregularity of an arbitrary plane figure. To those who sought a natural world governed by reason and order, there was much appeal in the process of replacing the asymmetric by the symmetric, the imperfect by the perfect, the irrational by the rational. In this sense, quadrature represented not only the triumph of human reason, but also the inherent simplicity and beauty of the universe itself.

Devising quadratures was thus a particularly fascinating problem for Greek mathematicians, and they produced clever geometric constructions to that end. As is often the case in mathematics, solutions can be approached in stages, by first squaring a reasonably "tame" figure and moving from there to the quadrature of more irregular, bizarre ones. The key initial step in this process is the quadrature of the rectangle, the procedure for which appears as Proposition 14 of Book II of Euclid's *Elements*, although it was surely known well before Euclid. We begin with this.

STEP 1 Quadrature of the rectangle (Figure 1.7)

Let *BCDE* be an arbitrary rectangle. We must construct, with compass and straightedge only, a square having area equal to that of *BCDE*. With the straightedge, extend line *BE* to the right, and use the compass to mark off segment *EF* with length equal to that of *ED*—that is, $\overline{EF} = \overline{ED}$. Next, bisect *BF* at *G* (an easy compass and straightedge construction), and with center *G* and radius $\overline{BG} = \overline{FG}$, describe a semicircle as shown. Finally, at *E*, construct line *EH* perpendicular to *BF*, where *H* is the point of intersection of the perpendicular and the semicircle, and from there construct square *EKLH*.

We now claim that the shaded square having side of length $\overline{EH}$—a figure we have just *constructed*—has area equal to that of the original rectangle *BCDE*.

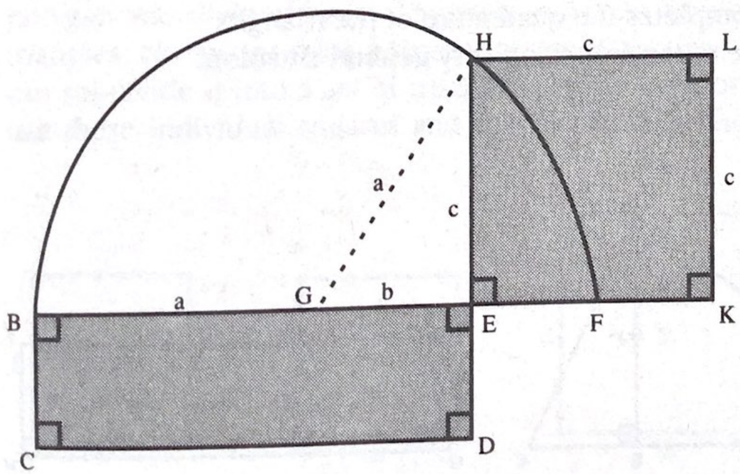

FIGURE 1.7

To verify this claim requires a bit of effort. For notational convenience, let a, b, and c be the lengths of segments HG, EG, and EH, respectively. Since $\triangle GEH$ is a right triangle by construction, the Pythagorean theorem gives us $a^2 = b^2 + c^2$, or equivalently $a^2 - b^2 = c^2$. Now clearly $\overline{FG} = \overline{BG} = \overline{HG} = a$, since all are radii of the semicircle. Thus, $\overline{EF} = \overline{FG} - \overline{EG} = a - b$ and $\overline{BE} = \overline{BG} + \overline{GE} = a + b$. It follows that

$$\begin{aligned}\text{Area (rectangle } BCDE) &= (\text{base}) \times (\text{height})\\ &= (\overline{BE}) \times (\overline{ED})\\ &= (\overline{BE}) \times (\overline{EF}), \text{ since we constructed } \overline{EF} = \overline{ED}\\ &= (a+b)(a-b) \text{ by the observations above}\\ &= a^2 - b^2\\ &= c^2 = \text{Area (square } EKLH)\end{aligned}$$

Consequently, we have proved that the original rectangular area equals that of the shaded square which we *constructed* with compass and straightedge, and this completes the rectangle's quadrature.

With this done, the steps toward squaring more irregular regions come quickly.

STEP 2 Quadrature of the triangle (Figure 1.8)

Given $\triangle BCD$, construct a perpendicular from D meeting BC at point E. Of course, we call $\overline{DE}$ the triangle's "altitude" or "height" and know that the area of the triangle is ½(base) × (height) $= \frac{1}{2}(\overline{BC}) \times (\overline{DE})$. If we bisect DE at F and construct a rectangle with $\overline{GH} = \overline{BC}$ and $\overline{HJ} = \overline{EF}$, we know that the rectangle's area is $(\overline{HJ}) \times (\overline{GH}) = (\overline{EF}) \times (\overline{BC}) = \frac{1}{2}(\overline{DE}) \times (\overline{BC}) =$ area $(\triangle BCD)$. But we then apply Step 1 to construct a square equal in area to this rectangle, and so the square's area is also that of $\triangle BCD$. This completes the quadrature of the triangle.

We next move to the following very general situation.

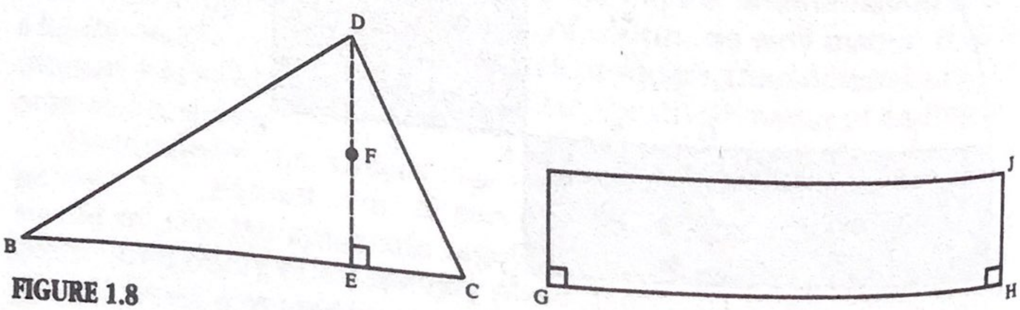

FIGURE 1.8

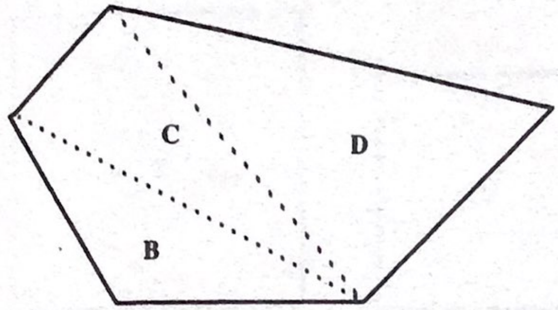

FIGURE 1.9

STEP 3 Quadrature of the polygon (Figure 1.9)

This time we begin with a general polygon, such as the one shown. By drawing diagonals, we subdivide it into a collection of triangles with areas **B**, **C**, and **D**, so that the total polygonal area is **B** + **C** + **D**.

Now triangles are known to be quadrable by Step 2, so we can construct squares with sides b, c, and d and areas **B**, **C**, and **D** (Figure 1.10). We then construct a right triangle with legs of length b and c, whose hypotenuse is of length x, where $x^2 = b^2 + c^2$. Next, we construct a right triangle with legs of length x and d and hypotenuse y, where we have $y^2 = x^2 + d^2$, and finally, the shaded square of side y (Figure 1.11).

Combining our facts, we see that

$$y^2 = x^2 + d^2 = (b^2 + c^2) + d^2 = \mathbf{B} + \mathbf{C} + \mathbf{D}$$

so that the area of the original polygon equals the area of the square having side y.

This procedure clearly could be adapted to the situation in which the polygon was divided by its diagonals into four, five, or any number of triangles. No matter what polygon we are given (see Figure 1.12), we can subdivide it into a set of triangles, square each one by Step 2, and use these individual squares and the Pythagorean theorem to build a

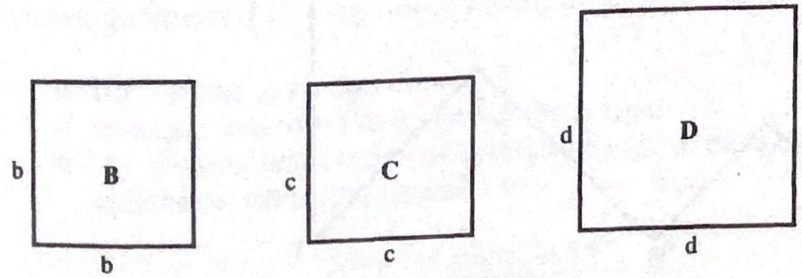

FIGURE 1.10

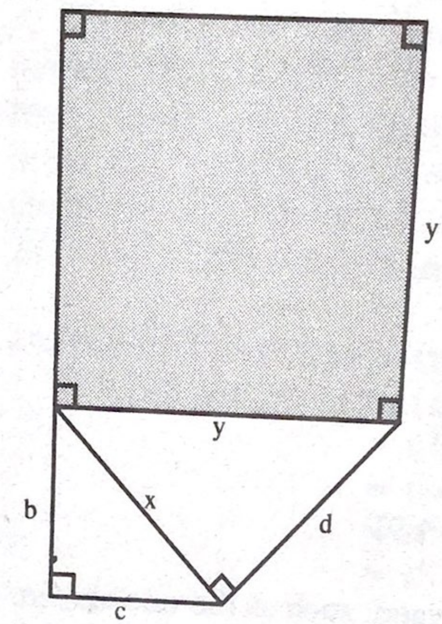

FIGURE 1.11

large square with area equal to that of the polygon. In short, polygons are quadrable.

By an analogous technique we could likewise square a figure whose area was the *difference* between—and not the sum of—two quadrable areas. That is, suppose we knew that area **E** was the difference between areas **F** and **G**, and we had already constructed squares of sides *f* and *g* with areas as shown in Figure 1.13. Then we would construct a right triangle with hypotenuse *f* and leg *g*. We let *e* be the length of the other leg and construct a square with side *e*. We then have

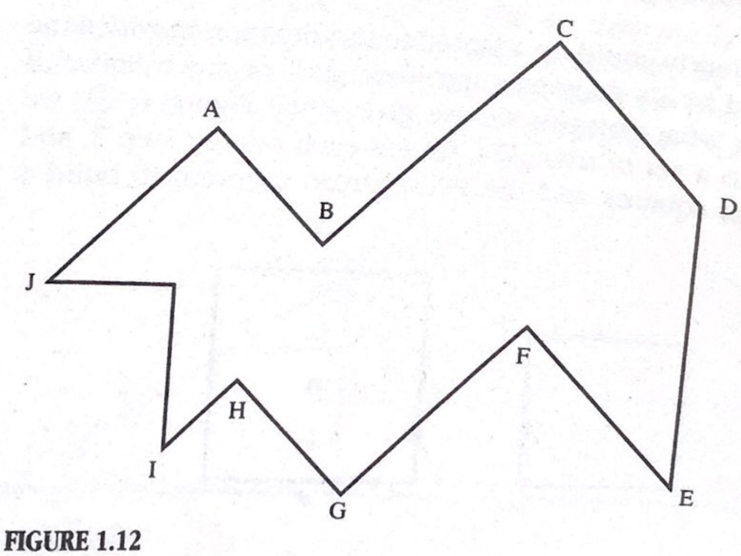

FIGURE 1.12

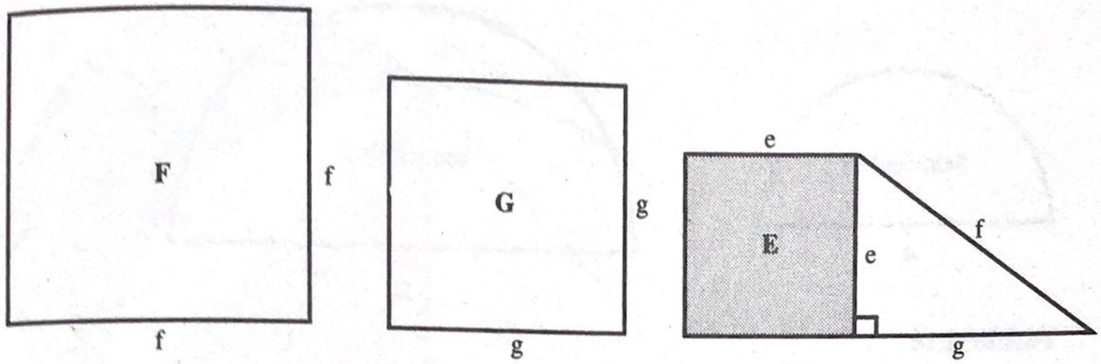

FIGURE 1.13

$$\text{Area (square)} = e^2 = f^2 - g^2 = \mathbf{F} - \mathbf{G} = \mathbf{E}$$

so that area **E** is likewise quadrable.

With the foregoing techniques, the Greeks of Hippocrates' day could square wildly irregular polygons. But this triumph was tempered by the fact that such figures are *rectilinear*—that is, their sides, although numerous and meeting at all sorts of strange angles, are merely straight lines. Far more challenging was the issue of whether figures with curved boundaries—the so-called *curvilinear* figures—were likewise quadrable. Initially, this must have seemed unlikely, for there is no obvious means to straighten out curved lines with compass and straightedge. It must therefore have been quite unexpected when Hippocrates of Chios succeeded in squaring a curvilinear figure known as a "lune" in the fifth century B.C.

Great Theorem: The Quadrature of the Lune

A lune is a plane figure bounded by two circular arcs—that is, a crescent. Hippocrates did not square all such figures but rather a particular lune he had carefully constructed. (As will be shown in the Epilogue, this distinction seemed to be the source of some misunderstanding in later Greek geometry.) His argument rested upon three preliminary results:

- The Pythagorean theorem
- An angle inscribed in a semicircle is right.
- The areas of two circles or semicircles are to each other as the squares on their diameters.

$$\frac{\text{Area (semicircle 1)}}{\text{Area (semicircle 2)}} = \frac{d^2}{D^2}$$

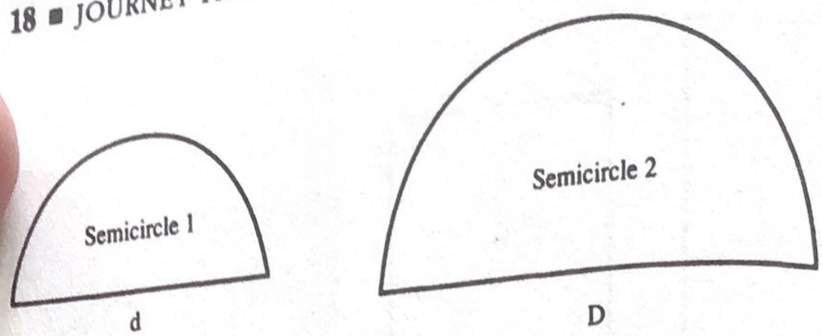

FIGURE 1.14

The first two of these results were well known long before Hippocrates came upon the scene. The last proposition, on the other hand, is considerably more sophisticated. It gives a comparison of the areas of two circles or semicircles based on the relative areas of the squares constructed on their diameters (see Figure 1.14). For instance, if one semicircle has five times the diameter of another, the former has 25 times the area of the latter. This proposition presents math historians with a problem, for there is widespread doubt that Hippocrates actually had a valid proof. He may well have *thought* he could prove it, but modern scholars generally feel that this theorem—which later appeared as the second proposition in Book XII of Euclid's *Elements*—presented logical difficulties far beyond what Hippocrates would have been able to handle. (A derivation of this result is presented in Chapter 4.)

That aside, we now consider Hippocrates' proof. Begin with a semicircle having center O and radius $\overline{AO} = \overline{OB}$, as shown in Figure 1.15. Construct OC perpendicular to AB, with point C on the semicircle, and draw lines AC and BC. Bisect AC at D, and using $\overline{AD}$ as a radius and D as center, draw semicircle AEC, thus creating lune $AECF$, which is shaded in the diagram.

Hippocrates' plan of attack was simple yet brilliant. He first had to establish that the lune in question had *precisely* the same area as the shaded ΔAOC. With this behind him, he could then apply the known fact that triangles can be squared to conclude that the lune can be squared as well. The details of the classic argument follow:

THEOREM Lune *AECF* is quadrable.

PROOF Note that $\angle ACB$ is right since it is inscribed in a semicircle. Triangles AOC and BOC are congruent by the "side-angle-side" congruence scheme, and consequently $\overline{AC} = \overline{BC}$. We thus apply the Pythagorean theorem to get

$$(\overline{AB})^2 = (\overline{AC})^2 + (\overline{BC})^2 = 2(\overline{AC})^2$$

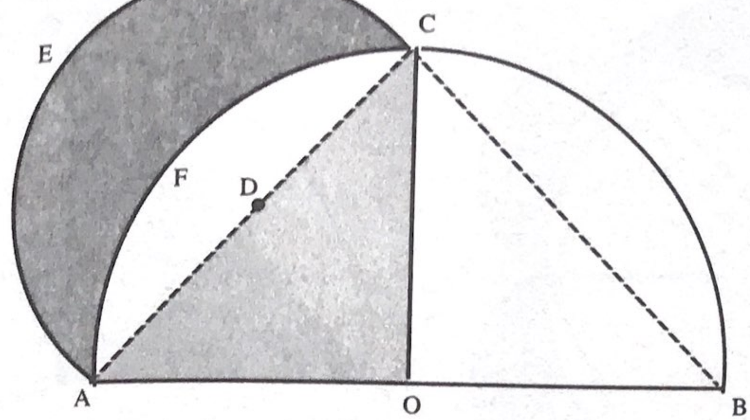

FIGURE 1.15

Because AB is the diameter of semicircle ACB, and AC is the diameter of semicircle AEC, we can apply the third principle above to get

$$\frac{\text{Area (semicircle } AEC)}{\text{Area (semicircle } ACB)} = \frac{(\overline{AC})^2}{(\overline{AB})^2} = \frac{(\overline{AC})^2}{2(\overline{AC})^2} = \frac{1}{2}$$

In other words, semicircle AEC has half the area of semicircle ACB.

But we now look at quadrant $AFCO$ (a "quadrant" is a quarter of a circle). Clearly this quadrant also has half the area of semicircle ACB, and we immediately conclude that

$$\text{Area (semicircle } AEC) = \text{Area (quadrant } AFCO)$$

Finally, we need only subtract from each of these figures their shared region $AFCD$, as in Figure 1.16. This leaves

$$\begin{aligned}\text{Area (semicircle } AEC) &- \text{Area (region } AFCD) \\ &= \text{Area (quadrant } AFCO) - \text{Area (region } AFCD)\end{aligned}$$

and a quick look at the diagram verifies that this amounts to

$$\text{Area (lune } AECF) = \text{Area } (\triangle ACO)$$

But, as we have seen, we can construct a square whose area equals that of the triangle, and thus equals that of the lune as well. This is the quadrature we sought.

Q.E.D.

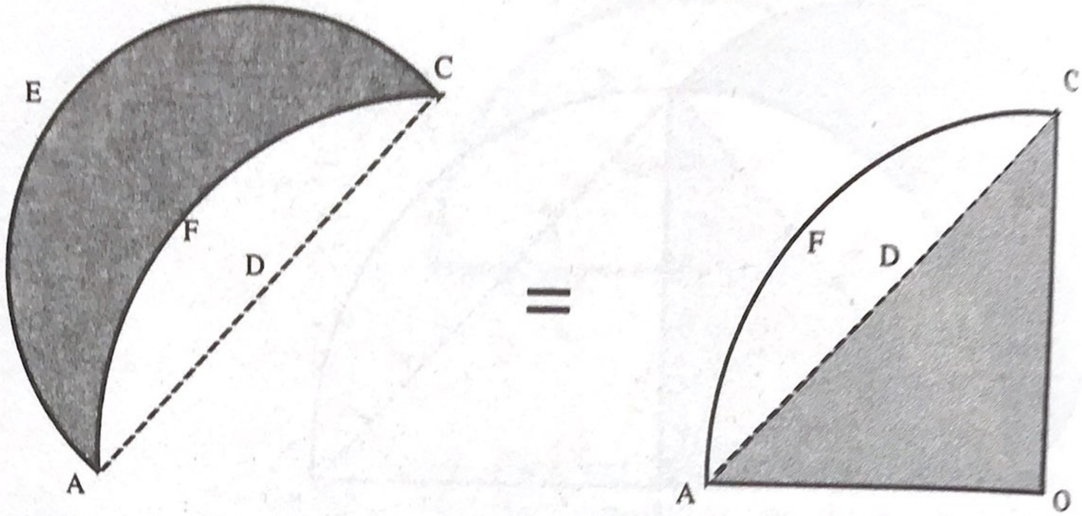

FIGURE 1.16

Here indeed was a mathematical triumph. Looking back from his fifth century vantage point, the commentator Proclus (A.D. 410–485) would write that Hippocrates of Chios ". . . squared the lune and made many other discoveries in geometry, being a man of genius when it came to constructions, if ever there was one."

Epilogue

With Hippocrates' success at squaring the lune, Greek mathematicians must have been optimistic about squaring that most perfect curvilinear figure, the circle. The ancients devoted much time to this problem, and some later writers attributed an attempt to Hippocrates himself, although the matter is again clouded by the difficulties of assessing commentaries upon commentaries. Nonetheless, Simplicius, writing in the fifth century, quoted his predecessor Alexander Aphrodisiensis (ca. A.D. 210) as saying that Hippocrates had claimed that he could square the circle. Piecing together the evidence, we gather that this is the sort of argument Alexander had in mind:

Begin with an arbitrary circle with diameter AB. Construct a large circle with center O and a diameter CD that is *twice* AB. Within the larger circle, inscribe a regular hexagon by the known technique of letting each side be the circle's radius. That is,

$$\overline{CE} = \overline{EF} = \overline{FD} = \overline{DG} = \overline{GH} = \overline{HC} = \overline{OC}$$

It is important to note that each of these segments, being the radius of the larger circle, also has length $\overline{AB}$. Then, using the six segments as diameters, construct the six semicircles shown in Figure 1.17. This gen-