Chapter 6

Cardano and the Solution of the Cubic

(1545)

A Horatio Algebra Story

Without question, the last decades of the fifteenth century marked a time of great intellectual excitement in Europe. Western civilization had clearly awakened from the slumber of the Middle Ages. Johannes Gutenberg had invented his marvelous printing press in 1450, and books became available as never before. Universities at Bologna, Paris, Oxford, and elsewhere had become legitimate centers of higher education and scholarship. In Italy, Raphael and Michaelangelo were beginning extraordinary artistic careers while their older countryman, Leonardo da Vinci, was giving meaning to the term Renaissance man.

It was not just the intellectual world whose horizons were expanding. In the year 1492, Christopher Columbus, a Genoa native, had discovered a new world far across the Atlantic Ocean. As much as anything, this discovery of the Americas stood as proof that contemporary civilization could extend the frontiers of knowledge beyond even the glorious legacy of the classical world. As the fifteenth century waned, there could be no doubt that Europe was on the threshold of great things.

And so it was in mathematics. In the year 1494, the Italian Luca Pacioli (ca. 1445–1509) produced a volume titled *Summa de Arithmetica*. In it, Pacioli treated the standard mathematics of his day, with emphasis on solving both linear and quadratic equations. Interestingly, he flirted with a primitive symbolic algebra by using *co* to denote the unknown quantity in his equations. This was short for *cosa*, the Italian word for "thing"—that is, the thing to be determined. It would be a century or more before algebra evolved into the symbolic system that we recognize today, but *Summa de Arithmetica* had taken a step in this direction.

Pacioli's assessment of the cubic equation—that is, an equation of the form $ax^3 + bx^2 + cx + d = 0$—was decidedly pessimistic. He had no idea how to solve the general cubic and expressed the belief that such a solution was as impossible, given the state of mathematics, as squaring the circle. This observation, actually something of a challenge laid before the Italian mathematical community, set the stage for the remarkable tale that surrounds our next great theorem: the sixteenth-century Italian algebraists and their quest for the solution of the cubic.

The story begins with Scipione del Ferro (1465–1526) of the University of Bologna. Taking up Pacioli's challenge, the talented del Ferro discovered a formula that solved the so-called "depressed cubic." This is a third-degree equation that lacks its second degree, or quadratic, term. That is, the depressed cubic looks like $ax^3 + cx + d = 0$. Usually, we prefer to divide through by *a* and move the constant term to the right-hand side of the equal sign, so as to convert the depressed cubic to its standard form

$$x^3 + mx = n$$

Renaissance Italians called this "cube and cosa equals number," for obvious reasons. Although he had mastered only this particular kind of cubic, del Ferro's algebraic advance was significant, and we would expect him to have spread the word of his triumph far and wide. Actually, he did nothing of the sort. The cubic's solution he kept an absolute secret!

To understand such behavior—almost incomprehensible in the "publish or perish" world of today—we must consider the nature of the Renaissance university. There, academic appointments were by no means secure. Along with patronage and political influence, continued service depended on the ability to prevail in public challenges that could be issued from any quarter at any time. Mathematicians like del Ferro always had to be ready to do scholarly battle with challengers, and the consequence of a public humiliation could be disastrous to one's career.

Thus, a major new discovery was a powerful weapon. Should an opponent appear with a list of problems to be solved, del Ferro could counter with a list of depressed cubics. Even if del Ferro were stumped by some of his challenger's problems, he could feel confident that his cubics, baffling to all but himself, would guarantee the downfall of his unfortunate adversary.

Scipione apparently did a good job of keeping his solution secret throughout his life, and it was only on his deathbed that he passed it along to his student Antonio Fior (ca. 1506–?). Although Fior was not so good a mathematician as his mentor, he rashly went on the offensive with his new-found weapon and in 1535 leveled a challenge at the noted Brescian scholar Niccolo Fontana (1499–1557).

An unfortunate childhood calamity had shaped Fontana's life. During the French attack on his home town in 1512, a soldier, sword in hand, had delivered a savage, slashing wound to the face of young Niccolo. According to legend, the boy survived only because a dog licked the horrible gash. But if the medicinal effects of canine saliva saved his life, they could not save his speech. So disfigured was Niccolo Fontana that he could no longer speak with clarity. Tartaglia—the Stammerer—became his nickname, and it is by this rather cruel epithet that he is best known today.

Physical deformities aside, Tartaglia was a gifted mathematician. In fact, he boasted that he could solve cubics of the form $x^3 + mx^2 = n$—that is, cubics missing their linear terms—although Fior doubted that Tartaglia had such a method. When the challenge from Fior arrived, Tartaglia sent him a list of 30 problems covering various mathematical topics. By contrast, Fior had provided a list of 30 "depressed cubics" and thereby placed Tartaglia in a bind. It was clearly a case of Fior's putting all his eggs into one basket; Tartaglia was either going to get a score of 0 or of 30 depending on whether or not he found the secret.

Not surprisingly, Tartaglia began a frantic, round-the-clock attack on the depressed cubic. His frustrations mounted as the days passed and the critical deadline approached. Then, on the night of February 13, 1535, with time almost exhausted, Tartaglia discovered the solution. His intense efforts had paid off. He now could solve all of Fior's problems with ease, while his less gifted challenger turned in a dismal performance of his own. In a great public triumph, Tartaglia prevailed brilliantly. His reward was to have been 30 lavish banquets provided by the hapless Fior, but Tartaglia, in a gesture of magnanimity, relieved his opponent of this commitment. The monetary savings to Fior must have been of little value as compared to the total disgrace he had suffered; he quietly faded from the picture.

But then entered perhaps the most bizarre character in the whole history of mathematics, Gerolamo Cardano (1501–1576) of Milan. Cardano had heard of the challenge and desired to learn more of the wonderful techniques of Tartaglia, the master of the cubic equation. Rather boldly, Cardano asked the Brescian to divulge the secret, and from there the story took unexpected and remarkable turns.

Before following it to its conclusion, however, we should pause to examine the extraordinary life of Gerolamo Cardano. We are fortunate to have a first-person account in his autobiography *De Vita Propria Liber (The Book of My Life)* written in 1575. This book is awash with Cardano's recollections, peeves, and superstitions, not to mention a wealth of extremely peculiar anecdotes. More than most autobiographies, this one must be regarded skeptically; even so, it gives us a revealing glimpse of his turbulent life.

Cardano began with a brief discussion of his forebears. His family tree may have included Pope Celestino IV, not to mention a distant cousin Angiolo, who, at the venerable age of eighty

> begot sons—infants feeble as if with their father's senility . . . The eldest of these sons has lived to be seventy, and I hear that some of his children became giants.

Then, in a chapter called "My Nativity," Cardano revealed that "although various abortive medicines—as I have heard—were tried in vain" he survived, only to be "literally torn from my mother's womb." This experience left him nearly dead, and a bath of warm wine was required to bring the infant Gerolamo back to life. It appears that Cardano may have been illegitimate, thus explaining his unwelcome arrival, and the associated stigma played a key role in his life's story.

With such a shaky start, it should come as no surprise that Cardano was plagued with infirmities throughout his life. In his autobiography, he never hesitated to describe these afflictions, often in complete if not disgusting detail. He told of violent heart palpitations, of fluids oozing from the stomach and chest, of ruptures and hemorrhoids, not to mention a disease characterized by "an extraordinary discharge of urine" yielding up to 100 ounces (nearly a gallon) per day. He recorded an intense fear of high places, as well as "of places where there is any report of mad dogs having been seen." He experienced years of sexual impotence, which lasted until just before his marriage (certainly an example of good timing). It was not unusual for Cardano to experience eight consecutive nights of insomnia; at such times there was little he could do but "get up, walk around the bed, and count to a thousand many times."

On those rare occasions when he was not suffering from one of his horrible ailments, Cardano would consciously inflict pain upon himself. He did so because "I considered that pleasure consisted in relief following severe pain" and, when not suffering physically, "a certain mental anguish overcomes me, so grievous that nothing could be more distressing." Consequently,

> I have hit upon a plan of biting my lips, of twisting my fingers, of pinching the skin of the tender muscles of my left arm until the tears come.

Cardano was saying, more or less, that these self-inflicted tortures were desirable because it felt so good when he stopped.

Fragile physical (and mental) health was not his only problem. After compiling an excellent record at the University of Padua on the way to becoming a physician, Cardano was refused permission to practice medicine in his home of Milan. This refusal may have been due to his reputed illegitimacy or to his grating and bizarre personality, but whatever its cause, it marked one of the low points in a life notable for its ups and downs.

Rejected by Milan, Cardano moved to the small town of Sacco, near Padua, where he practiced medicine in the bucolic, if somewhat limiting, confines of country life. One night in Sacco, he dreamt of a beautiful woman in white. As one who put great stock in the meaning of dreams, he was thus strongly affected when, some time later, he encountered a woman exactly matching his dream apparition. At first, the poor Cardano despaired at the impossibility of courting her:

> If I, a pauper, marry a wife who has no dot save a troop of dependent brothers and sisters, I'm done for! I can scarcely pay my expenses as it is! If I should attempt an abduction, or try to seduce her, there would be plenty to spy upon me.

Still, his love made marriage irresistible. In 1531, he married Lucia Bandarini, the woman of his dreams.

As this episode suggests, dreams, omens, and portents figured prominently throughout Cardano's life. He was an ardent astrologer, a wearer of amulets, and a seer of visions who predicted the future from thunderstorms. In addition, he often felt the presence of a protective spirit, or guardian angel, as he remarked in his autobiography:

> Attendant or guardian spirits . . . are recorded as having favored certain men constantly—Socrates, Plotinus, Synesius, Dio, Flavius Josephus—and I include myself. All, to be sure, lived happily save Socrates and me . . .

Apparently, he did not hesitate to carry on lively conversations with his attendant spirit. Says Oystein Ore, Cardano's twentieth century biographer, "In the face of such tales it is no wonder that some of his contemporaries believed that he was not in his right mind."

Another of his life-long interests was gambling. Cardano regularly indulged in games of chance, often earning substantial sums to supplement his income. Contritely, he acknowledged in his autobiography

> . . . as I was inordinately addicted to the chess-board and the dicing table, I know that I must rather be considered deserving of the severest censure. I gambled at both for many years; and not only every year, but—I say it with shame—every day.

Fortunately, Cardano subjected this vice to a scientific scrutiny. His resulting *Book on Games of Chance*, published posthumously in 1663, was the first serious treatise on the mathematics of probability.

And so, casting horoscopes, constantly gambling, beginning a family, Gerolamo Cardano spent the years from 1526 to 1532 in Sacco. But neither his pocketbook nor his ego could endure the small-town atmosphere for long, and by 1532 Cardano, with wife Lucia and son Giambattista, was back in Milan, still forbidden to practice medicine and ultimately consigned to the poorhouse.

Then, at last, fortune smiled upon him. Cardano began giving lectures on popular science that were especially well received by the educated and nobility. He wrote successful treatises on topics ranging from medicine to religion to mathematics. In particular, in 1536 he published an exposé attacking the corrupt and inadequate practices of Italian doctors. This work, not surprisingly, was detested by the medical community but embraced by the public, and Cardano could be kept from practicing medicine no longer. The College of Physicians in Milan grudgingly accepted him into their ranks in 1539, and soon he shot to the top of his profession. By mid-century, Cardano was perhaps the most famous and sought-after doctor in Europe, one who treated the Pope and even traveled to Scotland—a long and arduous journey in those days—to care for the Archbishop of St. Andrew's.

His days of triumph were not to last, for personal tragedies soon intervened. In 1546, his wife died at age 31, leaving Cardano with two sons and a daughter. Of these, the elder son, Giambattista, was Cardano's hope and joy. The boy proved quite bright, taking his medical degree in Pavia, and appeared to be following his father into a brilliant medical career.

But disaster struck in the form of a "wild woman" (Cardano's words). He related that, on the night of December 20, 1557, " . . . when the desire (to sleep) was about to overcome me, my bed suddenly seemed to tremble, and with it the whole bed-chamber." The next morning, Cardano's inquiries revealed that no other townsperson had felt this nocturnal quake, and Cardano took it as a very bad omen. No sooner had he reached this conclusion than his servant brought the unexpected news that Giambattista had married a woman "utterly without dowry or recommendation."

Indeed, the match proved to be an unfortunate one. Giambattista's wife bore three children, none of whom, she boasted, was Giambattista's. Such infidelity, openly flaunted, brought the young man to the breaking point. In retaliation, he prepared for her a cake laced with arsenic. It did its job all too well, and Giambattista was arrested for murder. Cardano's tireless efforts and great reputation were to no avail; his beloved son was convicted and beheaded in early April 1560.

"This was my supreme, my crowning misfortune," the grieving Cardano wrote. Despondent, he lost his friends, his career, and his zest for life. Moreover, his other son, Aldo, was himself turning into a criminal, and Cardano actually was "obliged to have him imprisoned more than once." Heartbreak seemed to follow heartbreak.

In 1562, he abandoned Milan, the city of his triumphs and tragedies, and accepted a position in medicine at the University of Bologna. With him he took Fazio, Giambattista's son. Between the old man and the boy there developed a strong and loving relationship that perhaps, in his waning years, gave Cardano some of the joy that his own offspring had not.

But the young boy and the new city did not bring tranquility into this stormy life. In 1570, Cardano was arrested and jailed on charges of heresy. At the time, of course, the Church in Italy had adopted a hard line against the unorthodoxies of the Reformation, and it certainly found no comfort in Cardano's casting the horoscope of Jesus or writing the book *In Praise of Nero* about the hated, anti-Christian Roman emperor.

Jailed and humiliated, the aging Cardano seemed to have met with his final disgrace. Yet, thanks to the testimonials of his illustrious friends and the leniency of the Church, Cardano soon got out of prison, went to Rome, and somehow wound up with a pension from the Pope! His was a "Horatio Algebra" story, if ever there was one. Thus resurrected, joined by his beloved grandson, Cardano spent his last years. Although an old man, he noted proudly in his autobiography that he still possessed "fourteen good teeth, and one which is rather weak; but it will last a long time, I think, for it still does its share." Cardano spent his last years in relative tranquility and died quietly, after a very full life, on September 20, 1576.

To the modern reader, Cardano remains a fascinating, if self-contradictory, character. He was incredibly prolific; his collected works fill seven thousand pages and cover a bewildering array of topics, scientific and otherwise. Yet even as he had one foot planted in the modern, rational world, he had another squarely planted in the superstitious irrationality of the Middle Ages. Looking back a century later, the great philosopher and mathematician Gottfried Wilhelm Liebniz summed him up quite aptly: "Cardano was a great man with all his faults; without them he would have been incomparable."

We now return to the story of the cubic equation, in which Cardano was to play a major role. Recall that, in 1535, Tartaglia of Brescia had soundly bested Antonio Fior by discovering the solution of certain kinds of cubics. Cardano was intrigued. Again and again he wrote to Tartaglia begging for the solution, and again and again he was rebuffed, with Tartaglia vowing to write a book on the matter in his own good time. Initially, Cardano reacted with anger, but eventually more soothing words brought Tartaglia to Milan as Cardano's guest. There, on March 25, 1539, Tartaglia revealed the secret of the depressed cubic—albeit written in cipher—to Cardano, who took the following solemn oath:

> I swear to you by the Sacred Gospel, and on my faith as a gentleman, not only never to publish your discoveries, if you tell them to me, but I also promise and pledge my faith as a true Christian to put them down in cipher so that after my death no one shall be able to understand them.

A final character then appeared in this amazing drama. This was the young Ludovico Ferrari (1522–1565), who arrived at Cardano's door asking for work. Cardano had that very day perceived a good omen in the incessant squawking of a magpie and thus eagerly took the boy in as a servant. It soon became clear that the young Ludovico was extraordinarily precocious. Their relationship quickly turned from master/servant to teacher/pupil and eventually, before Ferrari was 20 years old, to colleague/colleague. Cardano shared Tartaglia's secret with his brilliant young protégé, and together the two of them made astounding progress.

For instance, Cardano discovered how to solve the general cubic equation

$$x^3 + bx^2 + cx + d = 0,$$

where the coefficients b, c, and d may or may not be zero. Unfortunately, Cardano's work rested upon reducing the general cubic to a depressed form and thus ran up against his pledge of secrecy to Tartaglia. Meanwhile, Ferrari succeeded in finding a technique for solving the quartic (or fourth degree) polynomial equation. This was a major discovery in algebra, but it depended upon reducing the quartic to a related cubic.

and again Cardano's oath forbade its publication. The two men, possessing the greatest algebraic discoveries of their time, were stymied.

But then, in 1543, Cardano and Ferrari traveled to Bologna where they inspected the papers of Scipione del Ferro, with whom this whole long story had begun nearly three decades earlier. There, in del Ferro's own hand, was the solution to the depressed cubic. To Cardano, the implication was clear: he no longer was prohibited from publishing this result, since it was from del Ferro, and not from Tartaglia, that he would take his cue. The fact that both solutions were identical did not particularly bother the eager Cardano.

And so, in the year 1545, there appeared Cardano's mathematical masterpiece *Ars Magna*. To him, algebra was the "Great Art," and the book represented a breathtaking advance over that which had been previously known. Its 40 chapters begin with simple algebraic matters, but it is in Chapter XI, titled "On the Cube and First Power Equal to the Number," that the world at last saw the solution of the cubic. It is worth noting that Cardano prefaced this key chapter with the following:

> Scipio Ferro of Bologna well-nigh thirty years ago discovered this rule and handed it on to Antonio Maria Fior of Venice, whose contest with Niccolo Tartaglia of Brescia gave Niccolo occasion to discover it. He gave it to me in response to my entreaties, though withholding the demonstration. Armed with this assistance, I sought out its demonstration in [various] forms. This was very difficult.

Cardano had thus given credit where credit was due, which satisfied everyone except Tartaglia. He, on the contrary, raged furiously about Cardano's deceit and treachery. In Tartaglia's eyes, Cardano had violated a sacred oath, pledged on his faith as a "true Christian," and was nothing more nor less than a vile scoundrel. Accusations poured from Tartaglia's pen and were answered not by Cardano, who managed to stay above the fray, but by the tenacious and loyal Ferrari. The latter was known for his hot temper (he had lost a few fingers in an especially vicious fight) and lashed back vehemently. Accusatory, volatile letters flew between Brescia and Milan. For instance, in a 1547 broadside, Ferrari blasted Tartaglia as

> . . . someone who spends the whole time . . . on trifles. I promise you that if it were up to me to reward you, I would load you up so much with roots and radishes that you would never eat anything else in your life.

(The last sentence is a pun on the mathematical roots that permeate cubic problems.)

The conflict culminated in yet another public debate, this one between Tartaglia and Ferrari in Milan on August 10, 1548. Tartaglia later made much of Cardano's absence, blaming him for a cowardly decision "to avoid being present at the dispute." However, the contest, held on Ferrari's home turf, proved a failure for the visitor. Tartaglia blamed this on the rowdiness and partisanship of the crowd, whereas Ferrari naturally attributed the outcome to his own intellectual superiority. In any case, Tartaglia withdrew to return home, and Ferrari was proclaimed the brilliant victor. Mathematics historian Howard Eves, noting the hostile crowd and Ferrari's hot-headed reputation, says that Tartaglia may have been fortunate to escape alive.

These, then, were the events surrounding the solution of the cubic, a story at once complex, lusty, and absurd. It now remains for us to consider the great theorem at the heart of this strange tale.

Great Theorem: The Solution of the Cubic

Upon examining Chapter XI of *Ars Magna*, the modern reader has two surprises in store. One is that Cardano gave not a general proof but a specific example of a depressed cubic, namely

$$x^3 + 6x = 20$$

although in our discussion below we shall treat the more general

$$x^3 + mx = n$$

The second is that his argument was purely geometrical, involving literal cubes and their volumes. Actually, the surprise here is minimized when we recall the primitive state of algebraic symbolism and the exalted position of Greek geometry among Renaissance mathematicians.

The key result of Chapter XI is stated here in Cardano's own words, and his clever dissection of the cube is presented. His wordy "rule" for solving cubics at first sounds quite confusing, but recasting it in a more familiar, algebraic light shows that it does the job.

THEOREM Rule to solve $x^3 + mx = n$:

> Cube one-third the coefficient of x; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate [repeat] this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same. Then, subtracting the cube root of the first from the cube root of the second, the remainder which is left is the value of x.

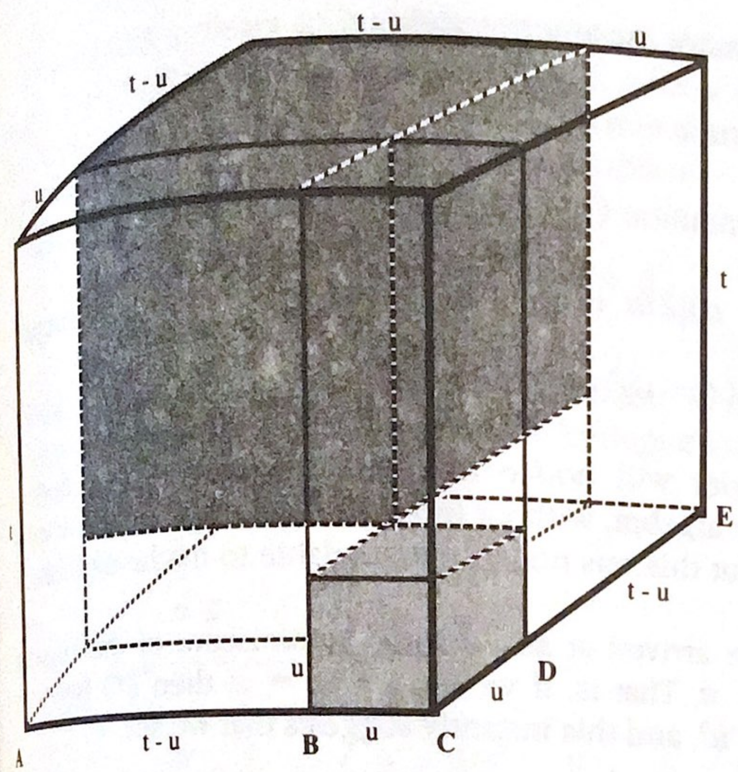


FIGURE 6.1

PROOF Cardano imagined a large cube, having side AC, whose length we shall denote by t, as shown in Figure 6.1. Side AC is divided at B into segment BC of length u and segment AB of length $t - u$. Here t and u are serving as auxiliary variables whose values we must find. As the diagram suggests, the large cube can be sliced into six pieces, each of whose volumes we now determine:

- a small cube in the lower front corner, with volume u^3
- a larger cube in the upper back corner, with volume $(t - u)^3$
- two upright slabs, one facing front along AB and the other facing to the right along DE, each with dimensions $t - u$ by u by t (the length of the side of the big cube) and thus each with volume $tu(t - u)$
- a tall block in the upper front corner, standing upon the small cube, with volume $u^2(t - u)$
- a flat block in the lower back corner, beneath the larger cube, with volume $u(t - u)^2$

Clearly the large cube's volume, t^3, equals the sum of these six component volumes. That is,

$$t^3 = u^3 + (t - u)^3 + 2tu(t - u) + u^2(t - u) + u(t - u)^2$$

Some rearrangement of these terms yields

$$(t-u)^3 + [2tu(t-u) + u^2(t-u) + u(t-u)^2] = t^3 - u^3$$

and factoring the common $(t-u)$ from the bracketed expression gives

$$(t-u)^3 + (t-u)[2tu + u^2 + u(t-u)] = t^3 - u^3$$

or simply

$$(t-u)^3 + 3tu(t-u) = t^3 - u^3 \qquad (*)$$

(The modern reader will notice that this equation can be derived instantly by simple algebra, without recourse to the arcane geometry of cubes and slabs. But this was not a route available to mathematicians in 1545.)

In (*) we have arrived at an equation reminiscent of the original cubic $x^3 + mx = n$. That is, if we let $t - u = x$, then (*) becomes $x^3 + 3tux = t^3 - u^3$, and this instantly suggests that we set

$$3tu = m \quad \text{and} \quad t^3 - u^3 = n$$

If we now can determine the quantities t and u in terms of m and n from the original cubic, then $x = t - u$ will yield the solution we seek.

Ars Magna does not present a derivation of these quantities. Rather, Cardano simply provided the specific rule for solving the "Cube and Cosa Equal to the Number" that was cited above. Trying to decipher his purely verbal recipe is no easy feat and certainly makes one appreciate the concise, direct approach of a modern algebraic formula. Exactly what was Cardano saying in this passage?

To begin, consider his two conditions on t and u, namely

$$3tu = m \quad \text{and} \quad t^3 - u^3 = n$$

From the former, we see that $u = m/3t$, and substituting this into the latter yields

$$t^3 - \frac{m^3}{27t^3} = n$$

Multiply both sides by t^3 and rearrange terms to get the equation

$$t^6 - nt^3 - \frac{m^3}{27} = 0$$

At first, this appears to be no improvement whatever, for we have aded our original third-degree equation in x for a sixth-degree equaon in t. What saved the day, of course, was that the latter can be egarded as a *quadratic* equation in the variable t^3:

$$(t^3)^2 - n(t^3) - \frac{m^3}{27} = 0$$

he quadratic formula, which had been available to mathematicians for enturies and which we mentioned in the Epilogue to the previous chaper, then yielded:

$$t^3 = \frac{n \pm \sqrt{n^2 + \frac{4m^3}{27}}}{2}$$

$$= \frac{n}{2} \pm \frac{1}{2}\sqrt{n^2 + \frac{4m^3}{27}} = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}$$

Then, using only the positive square root, we have

$$t = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

Now, we also know that $u^3 = t^3 - n$, and so we conclude that

$$u^3 = \frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}} - n \quad \text{or}$$

$$u = \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

At last, we have the algebraic version of Cardano's rule for solving the depressed cubic $x^3 + mx = n$, namely

$$x = t - u$$

$$= \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

Q.E.D.

This expression is called a "solution by radicals" or an "algebraic solution" for the depressed cubic. That is, it involves only the original coefficients in the equation—that is, m and n—and the algebraic operations of addition, subtraction, multiplication, division, and extraction of roots, used only finitely often. A little study shows that this formula yields precisely the same result as Cardano's verbal "Rule" stated above.

Note that the key insight in Cardano's argument was to replace the solution of the cubic by the solution of a related quadratic equation (in t^3). He thus found a way to lower the problem by "one degree" and to move from the unfamiliar turf of cubics to the well-known realm of quadratics. This very clever process suggested a path to follow in attacking equations of the fourth, fifth, and higher degrees well.

As a concrete example, Cardano solved his prototype cubic $x^3 + 6x = 20$. According to his recipe, he first cubed a third of the coefficient of x to get $(\frac{1}{3} \times 6)^3 = 8$; next he squared half of the constant term (that is, half of 20) to get 100, and then added the 8, yielding a sum of 108 whose square root he took. To this he both added and subtracted half of the constant term, to get $10 + \sqrt{108}$ and $-10 + \sqrt{108}$, and finally his solution was the difference of cube roots of these two numbers:

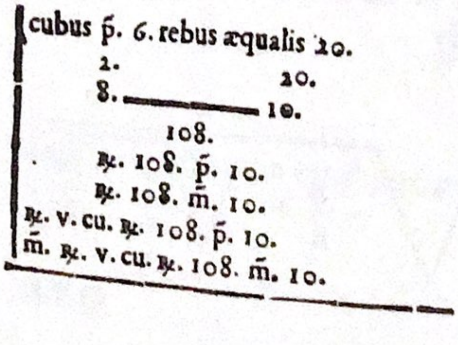
$$x = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$$

Of course, we could simply substitute $m = 6$ and $n = 20$ into the pertinent algebraic formula. This yields

$$\sqrt{\frac{n^2}{4} + \frac{m^3}{27}} = \sqrt{108} \qquad \text{and so}$$

$$x = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$$

which is clearly a "solution by radicals." It may come as a surprise—easily checked by a hand calculator—that this sophisticated-looking expression is nothing more than the number "2" in disguise, as Cardano


cubus p̄. 6. rebus æqualis 20.
2. 20.
8. ——— 10.
108.
℞. 108. p̄. 10.
℞. 108. m̄. 10.
℞. v. cu. ℞. 108. p̄. 10.
m̄. ℞. v. cu. ℞. 108. m̄. 10.

Cardano's Rule for the cubic, from *Ars Magna* (photograph courtesy of Johnson Reprint Corporation) correctly pointed out. One readily sees that $x = 2$ is indeed a solution of $x^3 + 6x = 20$.

Further Topics on Solving Equations

Observe that, having found one solution to the cubic, we are now in a position to find any others. For instance, since $x = 2$ solves the specific equation above, we know that $x - 2$ is one factor of $x^3 + 6x - 20$, and long division will generate the other, second-degree factor. In this case, $x^3 + 6x - 20 = (x - 2)(x^2 + 2x + 10)$. The solutions to the original cubic thus arise from solving the linear and quadratic equations

$$x - 2 = 0 \qquad \text{and} \qquad x^2 + 2x + 10 = 0$$

which is easily done. (This particular quadratic has no real solutions, so the cubic has as its only real solution $x = 2$.)

To the modern reader, the next two chapters of *Ars Magna* seem superfluous. Cardano titled Chapter XII "On the Cube Equal to the First Power and Number"—that is, $x^3 = mx + n$—and Chapter XIII was "On the Cube and Number Equal to the First Power"—that is, $x^3 + n = mx$. Today, we would regard these as having already been adequately covered by the formula above, for we would allow m and n to be negative. Mathematicians in the sixteenth century, however, demanded that all coefficients in the equation be positive. In other words, they regarded $x^3 + 6x = 20$ and $x^3 + 20 = 6x$ not just as different equations, but as intrinsically different ***kinds*** of equations. Such squeamishness about negative numbers is hardly surprising, given Cardano's tendency to think in terms of three-dimensional cubes, where sides of negative length make no sense. Of course, avoiding negatives led to a proliferation of cases and made *Ars Magna* considerably longer than we now find necessary.

So, Cardano could solve the depressed cubic in any of its three versions. But what about the ***general*** third-degree equation of the form $ax^3 + bx^2 + cx + d = 0$? It was Cardano's great discovery that, by means of a suitable substitution, this equation could be replaced by a related, depressed cubic that was, of course, susceptible to his formula. Before examining this "depressing" process for the cubic, we might take a quick look at it in a more familiar setting—as applied to solving quadratic equations:

Suppose we begin with the general second-degree equation

$$ax^2 + bx + c = 0 \qquad \text{where } a \neq 0$$