MA 565 Homework 7

Due Friday, October 16

1. Let k be a field. Show that the dot product $\cdot : k^n \times k^n \to k$, given by

$$(x_1, \dots x_n) \cdot (y_1, \dots, y_n) = \sum_{i=1}^{n} x_i y_i,$$

is bilinear.

2. Show that the cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, given by

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1),$$

is bilinear.

3. Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are not isomorphic as vector spaces over $\mathbb{R}$.

4. Let $\{e_1, e_2\}$ be a basis for $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

5. Let V be a vector space over a field k and let v, v' be nonzero elements of V. Prove that $v \otimes v' = v' \otimes v$ if and only if $v = av'$ for some $a \in k$.

6. Let V be a vector space over a field k. Show that $V \otimes_k k \cong V$.

7. Let U, V, and W be vector spaces over a field k. Show that

$$V \otimes (U \times W) \cong (V \otimes U) \times (V \otimes W).$$

8. Let U, V, and W be vector spaces over a field k. Show that

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W)).$$