MA 565 Homework 8

Due Friday, October 23

1. Let $T : U \to V$ be a linear map, and let W be a vector space. Show that the map $\bar{T} : U \otimes W \to V \otimes W$ given by
$$\bar{T}(\sum_{i=1}^{n} u_i \otimes w_i) = \sum_{i=1}^{n} T(u_i) \otimes w_i$$
is linear.

2. Let
$$0 \to V' \to V \to V'' \to 0$$
be a short exact sequence of vector spaces, and let W be a vector space. Show that
$$0 \to V' \otimes W \to V \otimes W \to V'' \otimes W \to 0$$
is also exact, where the maps are as in the previous problem.

3. Find an example of a vector space V, over a field k of characteristic 2, and a bilinear map $T : V \times V \to k$ that cannot be decomposed as a sum of symmetric and alternating bilinear maps.

4. Let $V = k[x]_{\leq 1}$ be the set of linear polynomials in one variable over a field k. Construct an isomorphism from $\mathrm{Sym}^d V$ to $k[x]_{\leq d}$.

5. Let V and W be vector spaces over a field k, $\mathrm{char} k \neq 2$, and let $T : V \times V \to W$ be a bilinear map. Prove that T is alternating if and only if $T(v, v) = 0$ for all $v \in V$.

6. The triple product of 3 vectors in $\mathbb{R}^3$ is the signed volume of the parallelopiped that they span, given by
$$T((u_1, u_2, u_3), (v_1, v_2, v_3), (w_1, w_2, w_3)) = \sum_{\sigma \in S_3} \mathrm{sign}(\sigma) u_{\sigma(1)} v_{\sigma(2)} w_{\sigma(3)}.$$
Show that $T : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is an alternating trilinear map.

7. Let V and W be vector spaces over a field k. Prove that
$$\bigwedge^n (V \oplus W) \cong \bigoplus_{p+q=n} \bigwedge^p V \otimes \bigwedge^q W.$$