MA 665 EXERCISES 10

(1) Show that a chain complex $C.$ is a projective object in **Ch** if and only if it is a split exact complex of projectives. Use this to show that if a category $\mathcal{C}$ has enough projectives, then so does $\mathbf{Ch}(\mathcal{C})$.

(2) Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor between two abelian categories, and let $G : \mathcal{B} \to \mathcal{C}$ be an exact functor. Prove that $G(L_iF) \cong L_i(GF)$.

(3) Show that the following are equivalent.
  (a) A is a projective R-module.
  (b) $\mathrm{Ext}^i_R(A, B) = 0$ for all $i \neq 0$ and all B.
  (c) $\mathrm{Ext}^1_R(A, B) = 0$ for all B.

(4) Show that the following are equivalent.
  (a) B is an injective R-module.
  (b) $\mathrm{Ext}^i_R(A, B) = 0$ for all $i \neq 0$ and all A.
  (c) $\mathrm{Ext}^1_R(A, B) = 0$ for all A.

(5) Let R be an integral domain with field of fractions K. Show that $\mathrm{Tor}^R_1(K/R, B)$ is the torsion submodule of B for every R-module B.

(6) Let p be a prime, suppose p^2 divides m, let $R = \mathbb{Z}/m\mathbb{Z}$ and $B = \mathbb{Z}/p\mathbb{Z}$. Show that

$$0 \to \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{p} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/p} \mathbb{Z}/m\mathbb{Z} \xrightarrow{p} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/p} \cdots$$

is an infinite periodic injective resolution of B. Prove that

$$\mathrm{Ext}^n_{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$$

for all n.