MA 665 EXERCISES 2

(1) Let A be a finite abelian group of order n and let p^k be the largest power of the prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.

(2) Let R be a commutative ring and let I, J be ideals of R.
   (a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 + I) \otimes (r + J)$.
   (b) Prove that there is an R-module isomorphism from $R/I \otimes_R R/J$ to $R/(I + J)$ mapping $(r + I) \otimes (r' + J)$ to $rr' + (I + J)$.

(3) Prove that extension of scalars from $\mathbb{Z}$ to the Gaussian integers $\mathbb{Z}[i]$ of the ring $\mathbb{R}$ is isomorphic to $\mathbb{C}$ as a ring. In other words, $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$ is isomorphic to $\mathbb{C}$ as a ring.