MA 665 EXERCISES 3

(1) Let S be a graded ring and $I \subseteq S$ a homogeneous ideal. Prove that

$$S/I = \bigoplus_{k=0}^{\infty} S_k/(I \cap S_k).$$

Conclude that S/I is a graded ring, and the quotient map $S \to S/I$ is a homomorphism of graded rings.

(2) Let R be an integral domain and let K be its field of fractions.
  (a) Considering K as an R-module, prove that $\wedge^2 K = 0$.
  (b) Let I be any R-submodule of K. Prove that $\wedge^k I$ is a torsion R-module for $k \geq 2$.
  (c) Give an example of an integral domain R and an R-submodule I of K such that $\wedge^k I \neq 0$ for all $k \geq 0$.

(3) Let K be a field and V a vector space over K.
  (a) Suppose that K has characteristic not equal to 2. Prove that a bilinear map $f(x, y)$ on V is alternating (that is, $f(x, x) = 0$ for all $x \in V$) if and only if $f(x, y) = -f(y, x)$ for all $x, y \in V$.
  (b) Now suppose that K has characteristic 2. Prove that every alternating bilinear map $f(x, y)$ on V is symmetric, but not every symmetric bilinear map is alternating.