CHIP FIRING EXERCISES 7

(1) Show that every bramble $\mathcal{B}$ on a chain of loops satisfies $\|\mathcal{B}\| \leq 3$. Conclude that the gonality of a graph can be arbitrarily higher than its treewidth.

(2) Let D be a divisor on a metric graph Γ, and let $\varphi, \psi \in \mathrm{PL}(\Gamma)$ be functions such that both $\mathrm{div}(\varphi) + D$ and $\mathrm{div}(\psi) + D$ are effective. Show that the pointwise minimum $\phi = \min\{\varphi, \psi\} \in \mathrm{PL}(\Gamma)$ also satisfies $\mathrm{div}(\phi) + D$ is effective.

(3) Find an example of a metric graph Γ and a divisor D on Γ such that $|D|$ is infinite, but D has rank zero.