CHIP FIRING

1. Divisors on Graphs

Throughout these notes, all graphs are assumed to be connected and loopless, though possibly with multi-edges. Our main object of study is divisors on graphs.

Definition 1.1. *A* divisor D *(or* chip configuration *or* sandpile*) on a graph G is a formal $\mathbb{Z}$-linear combination of vertices of G,*

$$D = \sum_{v \in V(G)} D(v) \cdot v$$

with $D(v) \in \mathbb{Z}$.

For example, Figure 1 depicts a divisor on the wedge of two triangles.

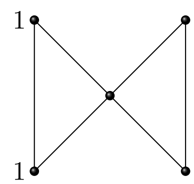

Figure 1. A divisor.

Divisors on graphs have been studied in combinatorics, computer science, and dynamics long before algebraic geometers got interested in them. In these disciplines it is more common to refer to divisors on graphs as *chip configurations* or *abelian sandpiles*. The term "chip configuration" comes from thinking of a divisor as a stack of poker chips on each vertex of the graph. Here we use the term *divisor* to emphasize the analogy with divisors on algebraic curves. Note that the divisors on a graph G form an abelian group, which we denote $\mathrm{Div}(G) = \mathbb{Z}^{V(G)}$.

We are interested in equivalence classes of divisors on graphs, where the equivalence is given by so-called chip-firing moves. Starting with a divisor, we may "fire" a vertex, which results in that vertex giving a chip to each of its neighbors. More concretely, we have the following definition.

Definition 1.2. *The* chip-firing move *at a vertex v takes a divisor D to D' where*

$$D'(w) = \begin{cases} D(v) - \mathrm{val}(v) & \text{if } w = v \\ D(v) + \#\text{ of edges between } w \text{ and } v & \text{if } w \neq v. \end{cases}$$

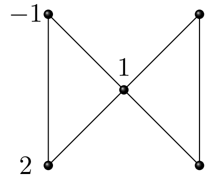

FIGURE 2. The result of firing a vertex.

In our example, if we fire the top left vertex, we get the divisor pictured in Figure 2.

If we fire a vertex v and then a vertex w, then on each edge connecting v and w, a chip travels from v to w and then from w to v, for a net change of zero. On each edge with only one endpoint at v or w, a chip travels away from this endpoint. Notice that this description does not depend on the order in which v and w are fired.

Definition 1.3. *Two divisors D, D' are* linearly equivalent*, and we write $D \sim D'$, if D' can be obtained from D by a sequence of chip-firing moves.*

Lemma 1.4. *Linear equivalence of divisors is an equivalence relation. Moreover, if $D_1 \sim D_2$ and $E_1 \sim E_2$ then $D_1 + E_1 \sim D_2 + E_2$.*

Proof. To see that linear equivalence is reflexive, note that any divisor is equivalent to itself by the empty sequence of chip-firing moves.

To see that linear equivalence is symmetric, it suffices to show that a chip-firing move can be inverted by a sequence of chip-firing moves. To see this, note that firing every vertex other than a given vertex v is the inverse of firing v.

For transitivity, suppose that $D_1 \sim D_2$ and $D_2 \sim D_3$. Then, by concatenating the sequence of chip-firing moves that takes D_1 to D_2 with that which takes D_2 to D_3, we obtain a sequence of chip-firing moves that takes D_1 to D_3. Hence, $D_1 \sim D_3$.

Finally, by concatenating the sequence of chip-firing moves that takes D_1 to D_2 with that which takes E_1 to E_2, we obtain a sequence of chip-firing moves that takes $D_1 + E_1$ to $D_2 + E_2$. □

Definition 1.5. *The* Picard group *of a graph G is the group of linear equivalence classes of divisors on G. That is,*

$$\mathrm{Pic}(G) = \mathrm{Div}(G)/\{\textit{divisors equivalent to } 0\}.$$

A divisor that is equivalent to 0 is called a principal divisor*.*

To compute the Picard group of a graph G, it helps to have an algebraic description of the principal divisors. This is accomplished by way of the Laplacian matrix.

Definition 1.6. *The* graph Laplacian *of a graph G is the square matrix with rows and columns indexed by the vertices of G, and whose (i, j)th entry is*

$$\Delta_{i,j} = \begin{cases} -\mathrm{val}(v_i) & \textit{if } i = j \\ \#\textit{ of edges between } v_i \textit{ and } v_j & \textit{if } i \neq j. \end{cases}$$

That is, Δ is the difference of the adjacency matrix and the valency matrix.

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Example 1.7. Consider the graph pictured in Figure 3[1].

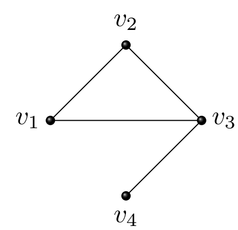

FIGURE 3. A simple graph.

The graph Laplacian of this graph is then

$$\Delta = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

If we consider a divisor as a vector in $\mathbb{Z}^{V(G)}$, then Δe_i is the divisor obtained from the identically zero divisor by firing the vertex v_i. More generally, for any vector $f \in \mathbb{Z}^{V(G)}$, Δf is the divisor obtained from 0 by firing each vertex v_i $f(i)$ times. Hence $\mathrm{Im}(\Delta)$ is exactly the set of divisors equivalent to 0. It follows that $\mathrm{Pic}(G) = \mathbb{Z}^{V(G)}/\mathrm{Im}(\Delta)$.

Note that $\det(\Delta) = 0$, because the sum of the columns of Δ is zero. From this we see that $\mathrm{Pic}(G)$ is infinite.

Example 1.8. By performing elementary row operations, we see that the Smith normal form of the matrix Δ from our previous example is

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, the Picard group of the graph G is $\mathrm{Pic}(G) \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

[1]In the film "Good Will Hunting", the first of the two problems to appear on the blackboard is a four-parter, the first part of which is to compute the Laplacian of this graph.