CHIP FIRING

10. Ramification

Definition 10.1. *Let D be a divisor of rank r on a graph G, and let v be a vertex of G. The sequence*

$$a_0 < a_1 < \cdots < a_r$$

defined by

$$a_i := \max\{m | \operatorname{rk}(D - mv) \geq r - i\}$$

is called the ramification sequence *of D at v. We say that v is a* ramification point *of D if the ramification sequence of D at v is anything other than $0 < 1 < \cdots < r$.*

Example 10.2. Let D be a divisor of degree $d \geq 0$ on a tree, and let v be any vertex of the tree. For $m \leq d+1$, we see that $\operatorname{rk}(D - mv) = \deg(D - mv) = d - m$. It follows that the ramification sequence of D at v is

$$0 < 1 < \cdots < d.$$

In other words, a divisor on a tree has no ramification points. This should be expected, because any two vertices of a tree are equivalent.

Example 10.3. Let D be a divisor of degree $d > 0$ on a cycle, and let v be a vertex of the cycle. For $m < d$, we see that $\operatorname{rk}(D - mv) = \deg(D - mv) - 1 = d - m - 1$. We therefore see that $a_i = i$ for all $i < d - 1$. Now, the rank of $D - dv$ is either 0 or -1. More precisely, $D - dv$ has rank 0 if and only if $D - dv \sim 0$. It follows that v is a ramification point of D if and only if $D \sim dv$.

Consider, for example, a cycle with 5 vertices. Label the vertices clockwise by $v_0, \ldots, v_4$. Recall that the map from $\operatorname{Jac}(G)$ to $\mathbb{Z}/5\mathbb{Z}$ given by

$$\sum_{i=0}^{4} a_i v_i \mapsto \sum_{i=0}^{4} i v_i \pmod 5$$

is an isomorphism. Because of this, any divisor of degree d is equivalent to a divisor of the form $(d-1)v_0 + v_i$ for some i. The vertex v_j is a ramification point of the divisor $D = (d-1)v_0 + v_i$ if and only if $i \equiv dj \pmod 5$.

If d is not divisible by 5, then there is a unique solution to this congruence. Thus, every divisor of degree d has a unique ramification point. On the other hand, if d is divisible by 5, then $dj \equiv 0 \pmod 5$. It follows that the divisor $D = (d-1)v_0 + v_i$ has no ramification points if $i \neq 0$, and every vertex is a ramification point of the divisor $D = dv_0$.

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Example 10.4. We consider ramification points of the canonical divisor on the graph pictured in Figure 1. Ramification points of the canonical divisor are typically referred to as *Weierstrass points*. Since the pictured graph has genus 2, the canonical divisor has rank 1. Moreover, for any vertex u of G, we have that

$$\mathrm{rk}(K_G - u) \leq \frac{1}{2}\deg(K_G - u) = \frac{1}{2},$$

so $\mathrm{rk}(K_G - u) = 0$. It follows that $a_0 = 0$.

Now, the divisor $K_G - 2v$ is v-reduced and not effective. The divisor $K_G - 2v$ therefore has negative rank. The ramification sequence of K_G at v is thus $a_0 = 0$, $a_1 = 1$, so v is not a Weierstrass point. On the other hand, the divisor $K_G - 2w$ is equivalent to the zero divisor. We therefore see that the ramification sequence of K_G at w is $a_0 = 0$, $a_1 = 2$, so w is a Weierstrass point.

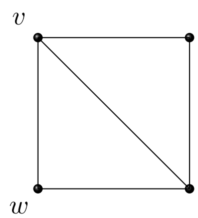

FIGURE 1. A graph of genus 2.

It is traditional to express the ramification sequence using partitions. In what follows, we identify the boxes in the Ferrers diagram of a partition with lattice points in $\mathbb{Z}^2_{>0}$[1].

Definition 10.5. *Let D be a divisor on a graph G of genus g, and let v be a vertex of G. We define the* Weierstrass partition *of D at v to be the partition*

$$\lambda_{G,v}(D) := \{(r+1, g-d+r) | \, \mathrm{rk}(D - (\deg(D) - d)v) \geq r\}.$$

We first note that the Weierstrass partition is indeed a partition. To see this, it suffices to show that if $(r+1, g-d+r) \in \lambda_{G,v}(D)$, then both $(r+1, g-d+r-1) \in \lambda_{G,v}(D)$ and $(r, g-d+r) \in \lambda_{G,v}(D)$. Let E be the divisor $D - (\deg(D) - d)v$. The first of these two implications follows from the fact that

$$\mathrm{rk}(E + v) \geq \mathrm{rk}(E) \text{ for any divisor } E.$$

The second follows from the fact that

$$\mathrm{rk}(E - v) \geq \mathrm{rk}(E) - 1 \text{ for any divisor } E.$$

We now record several other simple facts about Weierstrass partitions.

[1]We have chosen to depict partitions in the English style, which has the unfortunately reflects the y-axis from the standard coordinate system. In particular, the box $(1, 1)$ appears in the *upper* left of the Ferrers diagram, and the box $(1, 2)$ appears *below* it.

Lemma 10.6. *Let G be a graph of genus g and let v be any vertex of G. A divisor D on g has rank at least r if and only if*

$$(r+1, g-\deg(D)+r) \in \lambda_{G,v}(D).$$

Proof. By definition, we have $(r+1, g-\deg(D)+r) \in \lambda_{G,v}(D)$ if and only if

$$\mathrm{rk}(D - 0 \cdot v) \geq r.$$

□

One nice aspect of this definition is that the Weierstrass partition is invariant under addition of the vertex v.

Lemma 10.7. *Let G be a graph and let D be a divisor on G. For any vertex v of G, we have $\lambda_{G,v}(D) = \lambda_{G,v}(D+v)$.*

Proof. We have $(r+1, g-d+r) \in \lambda_{G,v}(D+v)$ if and only if

$$\mathrm{rk}(D + v - (\deg(D+v) - d)v) \geq r.$$

But

$$D + v - (\deg(D+v) - d)v = D - (\deg(D) - d)v,$$

and the result follows. □

The term $g-d+r$ in the definition of the Weierstrass partition may appear to be mysterious, but it is motivated by Riemann-Roch. There is a natural involution on the set of partitions given by the transpose. There is also a natural involution on the set of divisors given by mapping a divisor D to $K_G - D$. The Weierstrass partition is defined so that these two involutions agree.

Proposition 10.8. *Let G be a graph and v a vertex of G. For any divisor D on G, we have $\lambda_{G,v}(K_G - D) = \lambda^T_{G,v}(D)$.*

Proof. Suppose that $(r+1, g-d+r) \in \lambda_{G,v}(D)$. We must show that $(g-d+r, r+1) \in \lambda_{G,v}(K_G - D)$. By Riemann-Roch, we have

$$\begin{aligned}\mathrm{rk}(K_G - D - (\deg(K_G - D) - (2g-2-d))v) &= \mathrm{rk}(K_G - D - (d - \deg(D))v) \\ &= \mathrm{rk}(D - (\deg(D) - d)v) - d + g - 1 \geq g - d + r - 1.\end{aligned}$$

It follows that $(g-d+r, r+1) \in \lambda_{G,v}(K_G - D)$. □

Example 10.9. We again consider the canonical divisor in Example 10.4. We have $\mathrm{rk}(K_G) = 1$, so by Lemma 10.6, the Weierstrass partition $\lambda_{G,u}(K_G)$ contains the box $(2,1)$ for any vertex u. Recall that w is a Weierstrass point, but v is not. It follows that $\lambda_{G,w}(K_G)$ contains the box $(1,2)$, but $\lambda_{G,v}(K_G)$ does not. The Weierstrass partitions $\lambda_{G,w}(K_G)$ and $\lambda_{G,v}(K_G)$ are pictured in Figure 2.

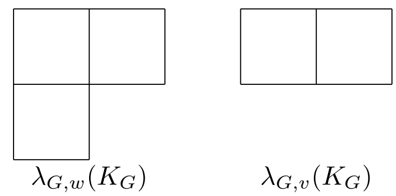

FIGURE 2. Weierstrass partitions for the canonical divisor on a graph of genus 2 at a Weierstrass and non-Weierstrass point.