CHIP FIRING

11. Extended Example: A Chain of Loops

In this lecture, we compute Weierstrass partitions (and therefore ranks) of divisors on a certain family of graphs. To begin, we consider the graph G pictured in Figure 1. Specifically, we let G' be a graph of genus $g-1$, and v a vertex of G. We let C be a cycle with m vertices, labeled counterclockwise by $v_0, \ldots, v_{m-1}$. We let G be the graph obtained by connecting the vertex v of G' to the vertex v_0 of C. Our goal is to compute the ranks of divisors on this graph G.

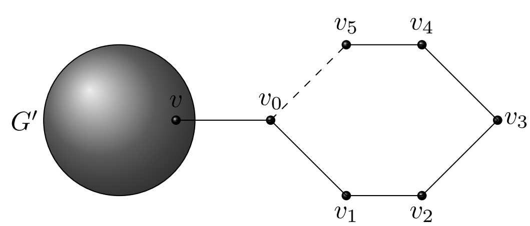

Figure 1. A graph with an attached cycle.

Lemma 11.1. *Let D be a divisor on G'. Then $\mathrm{rk}_G(D+v_i) \geq r$ if and only if:*

(1) $\mathrm{rk}_{G'}(D) \geq r$ *when* $i \neq 0$, *or*
(2) $\mathrm{rk}_{G'}(D+v) \geq r$ *and* $\mathrm{rk}(D-v) \geq r-1$, *when* $i = 0$.

Proof. First, suppose that $\mathrm{rk}_{G'}(D) \geq r$. Let E be an effective divisor of degree r on G, and let k be the degree of $E|_C$. Since D has rank at least r on G', we see that $D - kv - E|_{G'}$ is equivalent to an effective divisor. Because the degree of $kv_0 + v_i$ is $k+1 > 0$, we have $\mathrm{rk}_C(kv_0 + v_i) = k$, so $kv_0 + v_i - E|_C$ is equivalent to an effective divisor. It follows that

$$D + v_i - E \sim D + v_i - kv + kv_0 - E|_{G'} - E|_C$$

is equivalent to an effective divisor. Since E was arbitrary, we see that $\mathrm{rk}(D+v_i) \geq r$.

Second, suppose that $i = 0$, $\mathrm{rk}_{G'}(D+v) \geq r$, and $\mathrm{rk}(D-v) \geq r-1$. As above, let E be an effective divisor of degree r on G, and let k be the degree of $E|_C$. If $k = 0$, then since $D+v$ has rank at least r on G', we see that $D + v_0 - E \sim D + v - E$ is equivalent to an effective divisor. If $k > 0$, then since $D - v$ has rank at least $r-1$ on G', we see that $D - kv - E|_{G'}$ is equivalent to an effective divisor. As above, because

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the degree of $(k+1)v_0$ is $k+1>0$, we have $\text{rk}_C((k+1)v_0)=k$, so $(k+1)v_0-E|_C$ is equivalent to an effective divisor. It follows that

$$D+v_0-E\sim D-kv+(k+1)v_0-E|_{G'}-E|_C$$

is equivalent to an effective divisor. Since E was arbitrary, we see that $\text{rk}(D+v_0)\geq r$.

Finally, suppose that $\text{rk}(D+v_i)\geq r$. By definition, a divisor $D'+v_j$ on G is v-reduced if and only if D' is v-reduced on G' and $j\neq 0$. If $i\neq 0$, let E be an effective divisor of degree r on G', and let D' be the v-reduced divisor on G' equivalent to $D-E$. Since $i\neq 0$, $D'+v_i$ is v-reduced on G, and is therefore effective. But this implies that D' is effective. Since E was arbitrary, it follows that $\text{rk}_{G'}(D)\geq r$.

On the other hand, if $i=0$, let E be an effective divisor of degree r on G', and let D' be the v-reduced divisor on G' equivalent to $D+v-E$. The divisor D' is also v-reduced on G, and is therefore effective. But this implies that D' is effective. Since E was arbitrary, it follows that $\text{rk}_{G'}(D+v)\geq r$. Now, let E' be an effective divisor of degree $r-1$ on G', and let D' be the v-reduced divisor on G' equivalent to $D-v-E'$. For $j\neq 0$, the divisor $D'+v_j$ is v-reduced on G and equivalent to $D-E'-v_{m-j}$. It is therefore effective, hence D' is effective. Since E' was arbitrary, it follows that $\text{rk}(D-v)\geq r-1$. □

We now translate Lemma 11.1 into the language of Weierstrass partitions.

Proposition 11.2. *Let D be a divisor on G'. Then $\lambda_{G',v}(D)\subseteq\lambda_{G,v_j}(D+v_i)$. Moreover, a box $(x,y)\notin\lambda_{G',v}(D)$ is contained in the Weierstrass partition $\lambda_{G,v_j}(D+v_i)$ if and only if the following conditions hold:*

(1) $(x-1,y)\in\lambda_{G',v}(D)$,
(2) $(x,y-1)\in\lambda_{G',v}(D)$, *and*
(3) $i\equiv(\deg(D)-g-x+y)j\pmod{m}$.

Proof. By definition, we have $(r+1,g-d+r)\in\lambda_{G,v_j}(D+v_i)$ if and only if

$$\text{rk}(D+v_i+(d-\deg(D)-1)v_j)\geq r.$$

The divisor $D+v_i+(d-\deg(D)-1)v_j$ is equivalent to

$$D+(d-\deg(D)-1)v_0+v_k,$$

where

$$k\equiv i+(d-\deg(D)-1)j\pmod{m}\equiv i+(x-y+g-\deg(D))j\pmod{m}.$$

Thus, we have $(r+1,g-d+r)\in\lambda_{G,v_j}(D+v_i)$ if and only if

$$(*)\qquad \text{rk}(D+(d-\deg(D)-1)v+v_k)\geq r.$$

If $k\neq 0$, then by Lemma 11.1, $(*)$ holds if and only if

$$\text{rk}_{G'}(D+(d-\deg(D)-1)v)\geq r,$$

or, equivalently, we have $(r+1,(g-1)-(d-1)+r)\in\lambda_{G',v}(D)$.

If $k=0$, then by the above, we have

$$i\equiv(\deg(D)-g-x+y)j\pmod{m}.$$

By Lemma 11.1, $(*)$ holds if and only if

$$\text{rk}_{G'}(D+(d-\deg(D))v)\geq r\text{ and }\text{rk}_{G'}(D+(d-\deg(D)-2)v)\geq r-1.$$

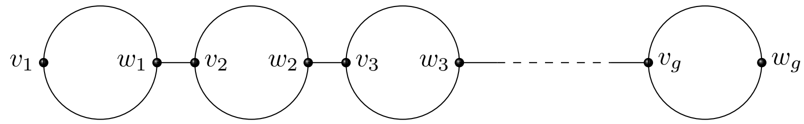

FIGURE 2. A chain of loops.

Equivalently, we have $(r+1,(g-1)-d+r),(r,(g-1)-(d-2)+(r-1) \in \lambda_{G',v}(D)$. $\square$

We now use Proposition 11.2 to compute the ranks of divisors on the graph pictured in Figure 2. We assume that the bottom part of each cycle is a single edge, while the top part of the kth cycle consists of $m_k - 1$ edges. (So the total number of edges in the kth cycle is m_k.) We define $\vec{m} = (m_1, \ldots, m_g)$, and we refer to this graph as the chain of g loops with torsion profile $\vec{m}$.

Let D be a divisor on this graph. In the previous lecture, we saw that $\lambda_{G,w_g}(D) = \lambda_{G,w_g}(D + (g - \deg(D))w_g)$, so we may assume that D has degree g. Every divisor of degree g is equivalent to a unique break divisor, so we may assume that D is a break divisor. In other words, D has exactly 1 "chip" on each cycle of G. That is, the restriction of D to any individual cycle in G has degree 1.

Let G_k be the union of the first k cycles of G, and for ease of notation, let $\lambda_k = \lambda_{G_k,w_k}(D|_{G_k})$. By Proposition 11.2, we have

$$\emptyset = \lambda_0 \subseteq \lambda_1 \subseteq \lambda_2 \subseteq \cdots \subseteq \lambda_g.$$

Moreover, a box $(x, y) \notin \lambda_{k-1}$ is contained in λ_k if and only if:

(1) $(x-1, y) \in \lambda_{k-1}$,
(2) $(x, y-1) \in \lambda_{k-1}$, and
(3) the distance from w_k to the chip of D on γ_k, in the counterclockwise direction, is equivalent to $y - x \pmod{m_k}$.

This sequence of partitions defines a tableau t on the partition λ_g, defined by

$$t(x, y) = k \text{ if } (x, y) \in \lambda_k \smallsetminus \lambda_{k-1}.$$

This tableau has the property that, if $t(x, y) = t(x', y') = k$, then $y - x \equiv y' - x' \pmod{m_k}$. Equivalently, the lattice distance between the boxes (x, y) and (x', y') is divisible by m_k. We say that a tableau with this property is an *$\vec{m}$-displacement tableau*.

Conversely, given an $\vec{m}$-displacement tableau t, we define

$$\lambda_k = \{(x, y) | t(x, y) \leq k\}.$$

We may then construct a break divisor D such that $\lambda_k \subseteq \lambda_{G_k,w_k}(D|_{G_k})$ for all k, as follows. If $t(x, y) = k$, then we place a chip on the kth loop, at a distance of $y - x \pmod{m_k}$ from w_k, in the counterclockwise direction. This is well-defined by the definition of $\vec{m}$-displacement tableaux. If the symbol k does not appear in the tableau t, then we place a chip at any vertex of the kth loop.

Lemma 11.3. *There exists a divisor of degree d and rank at least r on the chain of g loops with torsion profile $\vec{m}$ if and only if there exists an $\vec{m}$-displacement tableau* *with alphabet $\{1, \dots, g\}$ on the rectangular partition with $r+1$ columns and $g-d+r$ rows.*

Proof. This follows directly from the analysis above, combined with our observation from the previous lecture that a divisor D has rank at least r if and only if the Weierstrass partition $\lambda_{G,v}(D)$ contains the rectangle with $r+1$ columns and $g-\deg(D)+r$ rows. □

Corollary 11.4. *Suppose that $m_k \gg g$ for all k. Then there exists a divisor of degree d and rank r on the chain of g loops with torsion profile $\vec{m}$ if and only if*

$$g - (r+1)(g-d+r) \geq 0.$$

Proof. By Lemma 11.3, there exists a divisor of degree d and rank at least r on the chain of g loops with torsion profile $\vec{m}$ if and only if there exists an $\vec{m}$-displacement tableau on the rectangular partition with $r+1$ columns and $g-d+r$ rows. Because $m_k \gg g$ for all k, however, we see that no symbol can appear in such a tableau more than once. It follows that that there must be at least as many symbols as boxes in the rectangular partition. In other words, $g \geq (r+1)(g-d+r)$. □

Example 11.5. Let G be a chain of 4 loops, and suppose that $m_k \neq 2$ for all k. We consider divisors of degree 3 and rank 1 on G. By Lemma 11.3, such divisors exist if and only if there exists a standard Young tableau on the rectangular partition with 2 rows and 2 columns. There are 2 such tableaux, pictured in Figure 3.

Each of these tableaux corresponds to a divisor class of degree 3 and rank 1 on G. These 2 divisors are depicted in Figure 4. In this figure, the chips on the top of the loops are 1 edge away from the righthand vertex. We leave it to the reader to independently verify that these two divisors have rank 1. Note that, because one of the tableaux is the transpose of the other, the two illustrated divisors are Serre dual. In other words, if D is one of the pictured divisors, then the other one is $K_G - D$.

We now consider divisors of degree 2 and rank 1 on G. By Lemma 11.3, such divisors exist if and only if there is an $\vec{m}$-displacement tableau on a rectangular partition with 3 rows and 2 columns. The only tableau on a rectangle with these dimensions is picture in Figure 5. Because we assumed that $m_k \neq 2$ for all k, this is not an $\vec{m}$-displacement tableau. Thus, there does not exist a divisor of degree 2 and rank 1 on G. On the other hand, if we set $m_2 = m_3 = 2$, then there does exist such a divisor on G.

1	3
2	4

1	2
3	4

FIGURE 3. The two standard Young tableaux on a 2×2 rectangle.

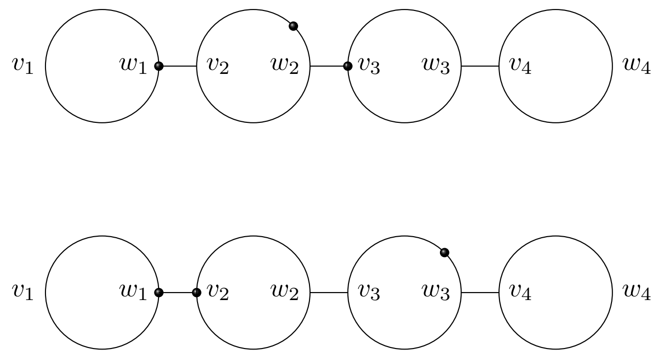

FIGURE 4. Two divisors of degree 3 and rank 1 on a chain of 4 loops.

1	2
2	3
3	4

FIGURE 5. The only tableau with 4 symbols on a 2×3 rectangle.