CHIP FIRING

13. Divisor Theory of Metric Graphs

We now turn from discrete graphs to metric graphs.

Definition 13.1. *A* metric graph *is a compact, connected metric space* Γ *obtained by identifying the edges of a graph* G *with line segments of fixed positive real length. The graph* G *is called a* model *for* Γ

Example 13.2. If we assign lengths to the edges of a cycle, we obtain a circle. Thus, the circle is a metric graph.

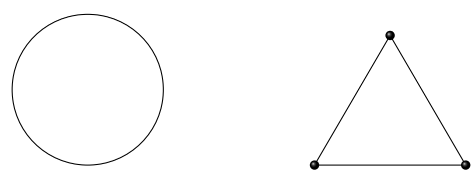

Figure 1. A metric graph and one of its models

Remark 13.3. A metric graph Γ does not have a unique model. Two graphs are models for the same metric graph if and only if they admit a common refinement.

Definition 13.4. *The* divisor group $\mathrm{Div}(\Gamma)$ *of a metric graph* Γ *is the free abelian group on points of the metric space* Γ.

Many properties of divisors can be defined in a way that is completely analogous to the discrete graph case.

Definition 13.5. *A divisor* $D = \sum a_i v_i$ *on a metric graph is* effective *if* $a_i \geq 0$ *for all* i. *Its* degree *is defined to be*

$$\deg(D) := \sum a_i.$$

As in the case of discrete graphs, we want to talk about equivalence of divisors. For this, we need a notion of rational functions on metric graphs.

Definition 13.6. *A* rational function *on a metric graph* Γ *is a continuous, piecewise linear function* $\varphi : \Gamma \to \mathbb{R}$ *with integer slopes. We write* $\mathrm{PL}(\Gamma)$ *for the group of rational functions on* Γ.

Example 13.7. Figure 2 indicates the domains of linearity and slopes of a rational function φ on a circle. It therefore determines the rational function up to translation.

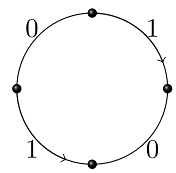

FIGURE 2. The domains of linearity and slopes of φ

Note that, in order for the function to be continuous, the two regions on which the function has slope 1 must be of equal length.

Definition 13.8. *Given $\varphi \in \mathrm{PL}(\Gamma)$ and $v \in \Gamma$, we define the* order of vanishing *of φ at v, $\mathrm{ord}_v(\varphi)$, to be the sum of the incoming slopes of φ at v. Note that $\mathrm{ord}_v(\varphi)$ is nonzero for only finitely many points $v \in \Gamma$. We define the divisor associated to φ to be*

$$\mathrm{div}(\varphi) = \sum_{v \in \Gamma} \mathrm{ord}_v(\varphi) \cdot v.$$

Divisors of the form $\mathrm{div}(\varphi)$ are called principal*.*

Example 13.9. The divisor associated to the rational function of Example 13.7 is pictured in Figure 3. Note that $\mathrm{div}(\varphi)$ is equal to $\mathrm{div}(\varphi + c)$ for any real number c. This is analogous to the fact that, on an algebraic curve, the divisor associated to a rational function is invariant under scaling the function by a non-zero constant. Indeed, we have the following.

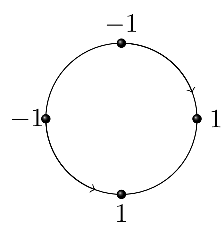

FIGURE 3. The principal divisor $\mathrm{div}\,\varphi$

Lemma 13.10. *Let Γ be a metric graph and let $\varphi, \psi \in \mathrm{PL}(\Gamma)$. We have $\mathrm{div}(\varphi) = \mathrm{div}(\psi)$ if and only if there exists a constant c such that $\varphi = \psi + c$.*

Proof. First, if $\varphi = \psi + c$, then φ and ψ have the same slope along every tangent vector of Γ. It follows that $\mathrm{ord}_v(\varphi) = \mathrm{ord}_v(\psi)$ for all $v \in \Gamma$, hence $\mathrm{div}(\varphi) = \mathrm{div}(\psi)$.

Conversely, if $\mathrm{div}(\varphi) = \mathrm{div}(\psi)$, then

$$\mathrm{div}(\varphi - \psi) = \mathrm{div}(\varphi) - \mathrm{div}(\psi) = 0.$$

It therefore suffices to show that, if $\phi \in \mathrm{PL}(\Gamma)$ satisfies $\mathrm{div}(\phi) = 0$, then ϕ is constant. To see this, consider the set $A \subseteq \Gamma$ where ϕ obtains its minimum. Since Γ is compact,

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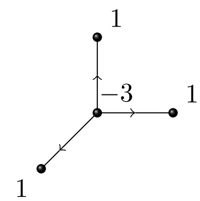

FIGURE 4. A rational function that is constant outside a local neighborhood, and its associated divisor

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the set A is nonempty. For any point v in the boundary of A, we have $\mathrm{ord}_v(\phi) < 0$. Thus, if $\mathrm{div}(\phi) = 0$, the boundary of A must be empty. Since Γ is connected, this implies that $A = \Gamma$, so ϕ is constant. □

Example 13.11. On any metric graph Γ, given a point $v \in \Gamma$, let $\epsilon \in \mathbb{R}$ be sufficiently small so that the open ball $B_\epsilon(v)$ contains no points of valence greater than 2 other than possibly v. Let χ be the rational function that takes the value ϵ on $\Gamma \smallsetminus B_\epsilon(v)$, the value 0 at v, and has slope 1 on the edges in $B_\epsilon(v)$ emanating from v. Then χ has order of vanishing $-\mathrm{val}(v)$ at v and 1 at each of the boundary points of $B_\epsilon(v)$. In this way, we can view addition of $\mathrm{div}(\chi)$ as a continuous version of chip firing, where we specify not only the vertex v that we fire from, but also the distance ϵ that we fire the chips.

We note the following.

Lemma 13.12. *The degree of a principal divisor is zero.*

Proof. Let $\varphi \in \mathrm{PL}(\Gamma)$, and let G be a model for Γ such that φ is linear on every edge of G. In other words, $V(G)$ contains the support of $\mathrm{div}(\varphi)$. For every edge e of G, let s_e denote the slope of φ along e. We see that the incoming slope of φ at one endpoint of e is s_e, and the incoming slope of φ at the other endpoint is $-s_e$. It follows that

$$\deg(\mathrm{div}(\varphi)) = \sum_{v \in V(G)} \mathrm{ord}_v(\varphi) = \sum_{e \in E(G)} [s_e - s_e] = 0.$$

□

Now that we have a notion of principal divisors on metric graphs, we can use it to define equivalence of divisors.

Definition 13.13. *We say that two divisors D and D' on a metric graph Γ are* equivalent *if $D - D'$ is principal. We define the* Picard group *of Γ to be the group of equivalence classes of divisors on Γ. That is,*

$$\mathrm{Pic}(\Gamma) = \mathrm{Div}(\Gamma)/\,\mathrm{div}(\mathrm{PL}(\Gamma)).$$

The Jacobian $\mathrm{Jac}(\Gamma)$ *of Γ is the group of equivalence classes of divisors of degree zero.*

The rank of a divisor on a metric graph is defined in exactly the same way as on a discrete graph. Since this definition is crucial to our interests, we record it here.

Definition 13.14. *Given* $D \in \mathrm{Div}(\Gamma)$*, the* complete linear series *of* D *is*

$$|D| := \{D' \sim D \mid D' \geq 0\}.$$

The rank *of* D *is the largest integer* r *such that* $|D - E| \neq \emptyset$ *for all effective divisors* E *of degree* r*.*

See the paper of Haase–Musiker–Yu for details on the structure of $|D|$.