CHIP FIRING

15. Riemann-Roch for Metric Graphs

Recall the statement of the Riemann-Roch Theorem.

Theorem 15.1 (Riemann-Roch for Metric Graphs)**.** *Let* Γ *be a metric graph of genus* g*. For any* $D \in \mathrm{Div}(\Gamma)$*,*

$$r(D) - r(K_\Gamma - D) = \deg(D) - g + 1.$$

Before proving this, we briefly describe the alternative argument of Gathmann and Kerber. First, one can use Riemann-Roch for discrete graphs to show that the result holds if Γ has rational edge lengths and D is supported on the $\mathbb{Q}$-points of Γ. One then uses the fact that such pairs (Γ, D) are dense in the set of all pairs.

Rather than reducing to Riemann-Roch for discrete graphs, we can instead mimic its proof. In their seminal paper, Baker and Norine describe a general strategy for proving theorems of Riemann-Roch type. Specifically, let X be a non-empty set and let $\mathrm{Div}(X)$ be the free abelian group on X. As usual, we define the degree of a divisor to be the sum of its coefficients, and we say that a divisor is effective if all of its coefficients are nonnegative. Let $\sim$ be an equivalence relation on X satisfying:

(1) if $D \sim D'$, then $\deg(D) = \deg(D')$, and
(2) if $D_1 \sim D_1'$ and $D_2 \sim D_2'$, then $D_1 + D_2 \sim D_1' + D_2'$.

We can define a rank function on $\mathrm{Div}(X)$ by declaring $\mathrm{rk}(D)$ to be -1 if D is not equivalent to an effective divisor, and otherwise declaring $\mathrm{rk}(D)$ to be the largest integer r such that $D - E$ is equivalent to an effective divisor for all effective divisors E of degree r. Finally, let g be an integer, and let $K \in \mathrm{Div}(X)$ be a divisor of degree $2g - 2$. Baker and Norine prove the following.

Theorem 15.2. *The Riemann-Roch formula*

$$r(D) - r(K_\Gamma - D) = \deg(D) - g + 1$$

holds for every $D \in \mathrm{Div}(X)$ *if and only if the following 2 conditions hold:*

(1) *For every* $D \in \mathrm{Div}(X)$*, either* $\mathrm{rk}(D) \geq 0$*, or there exists a divisor* D' *of degree* $g - 1$ *and rank* -1 *such that* $\mathrm{rk}(D' - D) \geq 0$*.*
(2) *For every* $D \in \mathrm{Div}(X)$ *of degree* $g - 1$*, if* $\mathrm{rk}(D) = -1$*, then* $\mathrm{rk}(K - D) = -1$*.*

It is straightforward to see that the 2 conditions follow from the Riemann-Roch formula. For the converse, we invite the reader to return to the proof of Riemann-Roch for graphs presented in Lecture 9, and see how the two conditions were used there.

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We now show that the 2 conditions hold for divisors on metric graphs. For the first condition, we must identify a suitably large collection of divisors of degree $g-1$ and rank -1. In the case of discrete graphs, these were the orientable divisors.

Definition 15.3. *An* orientation *on a metric graph* Γ *is an orientation of some model for* Γ*. As in the discrete case, given an orientation* $\mathcal{O}$*, define*

$$D_{\mathcal{O}} = \sum_{v\in\Gamma} (\mathrm{indeg}_{\mathcal{O}}(v) - 1)v.$$

Lemma 15.4. *If* $\mathcal{O}$ *is an acyclic orientation, then* $D_{\mathcal{O}}$ *is not equivalent to an effective divisor.*

Proof. Let $D = D_{\mathcal{O}} + \mathrm{div}(\varphi)$ for some $\varphi \in \mathrm{PL}(\Gamma)$, and let $A \subseteq \Gamma$ be the set where φ obtains its minimum. Since $\mathcal{O}$ is acyclic, there exists $v \in A$ such that v is a source in $\mathcal{O}|_A$. Then

$$D(v) \leq \mathrm{indeg}_{\mathcal{O}}(v) - \mathrm{outdeg}_A(v) - 1 < 0.$$

□

We now show that the first condition holds.

Lemma 15.5. *For any* $D \in \mathrm{Div}(\Gamma)$*, either* D *is equivalent to an effective divisor, or there exists an acyclic orientation* $\mathcal{O}$ *such that* $D_{\mathcal{O}} - D$ *is equivalent to an effective divisor. Moreover, for any* $v \in \Gamma$*,* $\mathcal{O}$ *can be taken to have unique source* v*.*

Proof. We may assume that D is v-reduced. Let G be a model for Γ with vertex set containing $\{v\} \cup \mathrm{Supp}(D)$. Run Dhar's burning algorithm on G and orient each edge in the direction that it burns. This produces an acyclic orientaiton $\mathcal{O}$ of G with unique source v. For any vertex $w \in V(G) \smallsetminus \{v\}$, since D is v-reduced we have $D(w) < \mathrm{indeg}_{\mathcal{O}}(w)$, so $D_{\mathcal{O}}(w) \geq D(w)$. Now, since D is v-reduced, D is equivalent to an effective divisor if and only if $D(v) > -1 = D_{\mathcal{O}}(v)$, and the result follows. □

The second condition follows from the previous two.

Lemma 15.6. *Let* $D \in \mathrm{Div}(\Gamma)$ *be a divisor of degree* $g-1$*. If* D *is not equivalent to an effective divisor, then* $K_\Gamma - D$ *is not equivalent to an effective divisor.*

Proof. If D is not equivalent to an effective divisor, then by Lemma 15.5, there exists an acyclic orientation $\mathcal{O}$ such that $D_{\mathcal{O}} - D$ is equivalent to an effective divisor. Since $\deg(D) = g-1$, the degree of $D_{\mathcal{O}} - D$ is zero. The only effective divisor of degree zero is the zero divisor, so $D \sim D_{\mathcal{O}}$. Now, if $\overline{\mathcal{O}}$ is the reverse orientation, then $D_{\mathcal{O}} + D_{\overline{\mathcal{O}}} = K_\Gamma$, so $K_\Gamma - D \sim D_{\overline{\mathcal{O}}}$. Since $\mathcal{O}$ is acyclic, $\overline{\mathcal{O}}$ is acyclic as well, and the result follows from Lemma 15.4. □

Combining Lemmas 15.5 and 15.6 with the theorem of Baker and Norine, we obtain the Riemann-Roch theorem for metric graphs.