CHIP FIRING

18. Rank Determining Sets

We begin with some definitions.

Definition 18.1. *Let D be a divisor on a metric graph Γ. We define the* complete linear series *of D to be*

$$|D| := \{D' \sim D \mid D' \textit{ is effective } \}.$$

In this language, a divisor D has rank at least r if, for every effective divisor E of degree r, the complete linear series $|D - E|$ is nonempty.

Definition 18.2. *Let D be a divisor on a metric graph Γ. We define the* support *of the complete linear series $|D|$ to be*

$$\mathrm{supp}(|D|) := \{v \in \Gamma \mid D'(v) > 0 \textit{ for some } D' \in |D|\}.$$

We say that a divisor D has support in A if $\mathrm{supp}(D)$ is contained in A.

On a discrete graph G, the rank of a divisor D can be computed as follows. Choose a vertex v of G. For each effective divisor E of degree r, run Dhar's Burning Algorithm to compute the v-reduced divisor equivalent to $D - E$. The divisor D has rank at least r if and only if this v-reduced is effective for all E.

On a metric graph, however, this procedure is impossible to implement because for $r > 0$ there are infinitely many effective divisors of degree r. The goal of this lecture is to show that there exists a finite set of "test" divisors E such that, if $|D - E|$ is nonempty for all E in this finite set, then the divisor D has rank at least r. This will make it feasible to compute the ranks of divisors on metric graphs. This idea is made precise by the notion of rank determining sets.

Definition 18.3. *Let Γ be a graph, and let A be a subset of Γ.*

1. *The A-rank $r_A(D)$ of a divisor D is the largest integer r such that $|D - E|$ is nonempty for all effective divisors E of degree r with support in A.*
2. *The set A is* rank determining *if $r_A(D) = r(D)$ for all $D \in \mathrm{Div}(G)$.*

Remark 18.4. Note that $r_A(D) \geq r(D)$ for any subset $A \subseteq \Gamma$ and any divisor D.

Definition 18.5. *Let $A \subseteq \Gamma$ be a subset. We define $\mathcal{L}(A)$ to be*

$$\mathcal{L}(A) = \bigcap_{\mathrm{supp}|D| \supseteq A} \mathrm{supp}|D|.$$

Proposition 18.6. *Let A be a nonempty subset of Γ. The following are equivalent:*

1. *$\mathcal{L}(A) = \Gamma$.*
2. *If D is a divisor with $r_A(D) \geq 1$, then $r(D) \geq 1$.*

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(3) *A is a rank-determining set.*

Proof. We first show that (1) and (2) are equivalent. To see this, note that $\mathcal{L}(A) = \Gamma$ if and only if, for any divisor D with $A \subseteq \text{supp}|D|$, we have $\text{supp}|D| = \Gamma$. Equivalently, for any such D, if $|D - v| \neq \emptyset$ for all $v \in A$, then $|D - v| \neq \emptyset$ for all $v \in \Gamma$. But this is the same as saying that $r_A(D) \geq 1$ implies $r(D) \geq 1$.

The implication (3) implies (2) is clear from the definition of rank determining set.

It remains to show that (2) implies (3). Assume (2). By Remark 18.4, it suffices to show that $r_A(D) \geq s$ implies $r(D) \geq s$ for all s. We proceed by induction. The case $s = -1$ is trivial, since $r(D) \geq -1$ for all D. The case $s = 0$ is also clear, since $r_A(D)$ and $r(D)$ are both nonnegative if and only if D is equivalent to an effective divisor.

We proceed by induction on s. Suppose $s \geq 1$ and $r_A(D) \geq s$. Then

$$r_A(D - v) \geq s - 1 \text{ for all } v \in A.$$

From the induction hypothesis, we deduce that $r(D-v) \geq s-1$. Therefore $D-E-v$ is equivalent to an effective divisor for all effective divisors E of degree $s-1$. Fixing E and varying v, this means that $r_A(D - E) \geq 1$. Having assumed (2), we deduce that $r(D - E) \geq 1$. Allowing E to vary over all effective divisors of degree $s-1$, we conclude that $D - E'$ is equivalent to an effective divisor for all effective divisors E' of degree s, and hence $r(D) \geq s$, as required. □

Knowing that A is a rank determining set if and only if $\mathcal{L}(A) = \Gamma$, we now provide a topological condition to determine when $\mathcal{L}(A) = \Gamma$. To do this we define a YL set.

Definition 18.7. *Let Γ be a graph and $U \subseteq \Gamma$ a connected open subset. We call U a* YL set *if every connected component X of the complement $\Gamma \smallsetminus U$ contains a point v such that* $\text{outdeg}_X(v) > 1$.

Remark 18.8. We have chosen to name YL sets after Ye Luo, whose work these notes are based upon.

We can characterize YL sets in terms of divisor theory.

Lemma 18.9. *Let $U \subseteq \Gamma$ be a nonempty connected open subset. Then U is a YL set if and only if $D = \sum_{v \in \partial U} v$ is w-reduced for any $w \in U$.*

Proof. This follows immediately by applying Dhar's burning algorithm, starting with any vertex $w \in U$. □

Given a divisor D on Γ, we may use Lemma 18.9 to find YL sets that are disjoint from the support of $|D|$.

Lemma 18.10. *For $v \in \Gamma$, let D be a v-reduced divisor, and let U be the set of vertices that can be reached from v by a path that does not pass through* $\text{supp}(D) \smallsetminus \{v\}$*. Then U is a YL set. Moreover, if $v \notin$* $\text{supp}D$*, then U is disjoint from* $\text{supp}|D|$*.*

Proof. Let $D' = \sum_{w \in \partial U} w$. Note that $D' \leq D$. Since D is v-reduced, we see that D' is v-reduced as well. By Lemma 18.9, it follows that U is a YL set. In addition, if $v \notin \text{supp}(D)$, then by Dhar's burning algorithm we see that D is w-reduced for all $w \in U$. It follows that $w \notin \text{supp}|D|$ for all $w \in U$. □

The following consequence of Lemma 18.10 is not necessary for our other results on rank-determining sets, but may be of independent interest.

Corollary 18.11. *Let D be a divisor on Γ. Then $(\mathrm{supp}|D|)^c$ is a disjoint union of YL sets.*

Proof. For each $v \in (\mathrm{supp}|D|)^c$, let D_v be the v-reduced divisor equivalent to D, and let $U_v \subseteq \Gamma$ be the set of points that can be reached from v by a path that does not pass through $\mathrm{supp}D_v$. By Lemma 18.10, U_v is a YL set disjoint from $\mathrm{supp}|D|$. It follows that

$$(\mathrm{supp}|D|)^c = \bigcup_{v \in (\mathrm{supp}|D|)^c} U_v$$

is a union of YL sets.

To see that this union is disjoint, suppose that $U_v \cap U_{v'} \neq \emptyset$. Then, since D_v is w-reduced for all $w \in U_v$ and $D_{v'}$ is w-reduced for all $w \in U_{v'}$, we see that $D_v = D_{v'}$ by uniqueness of reduced divisors. By the definition of U_v, we therefore see that $U_v = U_{v'}$. □

Remark 18.12. It is interesting to note that the disjoint union in Corollary 18.11 is a union of only finitely many disjoint YL sets. See Lemma 18.15 below.

We now turn to the main theorem of this lecture, which gives a sufficient condition for subsets of the vertices to be rank-determining.

Theorem 18.13. *Let $A \subseteq \Gamma$ be a nonempty subset. Then*

$$\mathcal{L}(A) \supseteq \bigcap_{\substack{U \, is \; YL \\ A \cap U = \emptyset}} U^c.$$

Moreover, if all YL sets intersect A, then A is a rank determining set.

Proof. First, suppose that $v \notin \mathcal{L}(A)$. By definition, there exists a divisor D such that $A \subseteq \mathrm{supp}|D|$ and $v \notin \mathrm{supp}|D|$. By Lemma 18.10, there exists a YL set U containing v that is disjoint from $\mathrm{supp}|D|$. Thus, v is not contained in the righthand side of the expression above.

To see the final remark, note that if all YL sets intersect A, then the righthand side of the expression above is Γ. Thus, in this case we have $\mathcal{L}(A) = \Gamma$, or equivalently, A is a rank determining set by Proposition 18.6. □

Remark 18.14. Luo proves the stronger result that the containment of Theorem 18.13 is in fact an equality, from which he derives that this sufficient condition for subsets to be rank-determining is also necessary. For our purposes, we will only need the fact that this condition is sufficient.

We note the following interesting property of YL sets.

Lemma 18.15. *If Γ is a metric graph of genus g and U is a YL set in Γ, then the closure $\overline{U}$ has genus at least 1. As a consequence, a collection of disjoint YL sets in Γ can contain at most g elements.*

Proof. If $\overline{U}$ is a tree, then every $v \in \partial U$ is a leaf of the tree, because otherwise U would be disconnected. It follows that, for every $v \in \partial U$, we have $\text{indeg}_U(v) = 1$, and thus U is not YL. □

The condition for rank determining sets provided in Theorem 18.13 is useful for many reasons. An important consequence of this result is that every metric graph contains a finite rank determining set.

Theorem 18.16. *Let Γ be a metric graph of genus g. Then there exists a rank-determining set of cardinality $g+1$.*

Proof. Let G be a model for Γ, let T be a spanning tree in G, and let $e_1, \ldots, e_g$ be the edges of G not in T. Choose a point $v_0 \in T$, and a point v_i in the interior of e_i for each i. We will show that the set $A = \{v_0, v_1, \ldots, v_g\}$ is a rank determining set.

By Theorem 18.13, it suffices to show that no YL set is contained in the complement of A. To see this, let $U \subset \Gamma \smallsetminus A$, and let X be the component of $\Gamma \smallsetminus U$ containing v_0. We will show that, if $v \in \partial X$, then $\text{outdeg}_X(v) = 1$. To see this, first suppose that v is contained in the interior of the edge e_i for some i. Since X contains a path from v_0 to v, we see that U cannot contain both endpoints of e_i. Since U is connected, it follows that $U \cap e_i$ is an interval with one endpoint at v. Therefore, $\text{outdeg}_X(v) = 1$. On the other hand, if $v \in T$, then for any $w \in U$, there exists a unique path from w to v. Since U is connected, it follows that $\text{outdeg}_X(v) = 1$. □

Remark 18.17. For some metric graphs there may exist rank determining sets of smaller cardinality that $g+1$. For example any three vertices form a rank determining set on the complete graph K_4, which has genus three.