CHIP FIRING

2. The Degree of a Divisor

In this lecture, we consider a fundamental invariant of divisors on graphs.

Definition 2.1. *The* degree *of a divisor* $D = \sum_{v \in V(G)} D(v)v$ *is the integer*

$$\deg(D) = \sum_{v \in V(G)} D(v).$$

Note that the degree is invariant under chip-firing. This allows us to make the following definition.

Definition 2.2. *The* Jacobian $\mathrm{Jac}(G)$ *of a graph* G *is the group of linear equivalence classes of divisors of degree 0 on* G*. (The Jacobian is also known as the* sandpile group*, or* critical group*, and probably many other things besides.)*

The degree is a group homomorphism $\mathrm{Pic}(G) \xrightarrow{\deg} \mathbb{Z}$. It is easy to see that this map is surjective, and the kernel is the group of divisors of degree 0. In other words, we have the short exact sequence

$$0 \to \mathrm{Pic}^0(G) \to \mathrm{Pic}(G) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Note that, since $\mathbb{Z}$ is free, the exact sequence above splits, so

$$\mathrm{Pic}(G) \cong \mathbb{Z} \oplus \mathrm{Jac}(G).$$

It follows that the degree d part of the Picard group, $\mathrm{Pic}^d(G)$, is a $\mathrm{Jac}(G)$-torsor. That is, the action of $\mathrm{Jac}(G)$ on $\mathrm{Pic}^d(G)$ by addition is free and transitive.

In the previous lecture, we saw that the Picard group of a graph can be computed using the graph Laplacian Δ. Note that $\det(\Delta) = 0$, because the sum of the columns of Δ is zero. From this we see that $\mathrm{Pic}(G)$ is infinite. Of course, this also follows from the fact that the degree homomorphism maps $\mathrm{Pic}(G)$ surjectively onto the integers. The order of the Jacobian is the absolute value of the determinant of the *reduced Laplacian*, which is the matrix $\widetilde{\Delta}$ obtained by removing any row from Δ and the corresponding column. More precisely, $\mathrm{Jac}(G) \cong \mathbb{Z}^{V(G)-1} / \mathrm{Im}(\widetilde{\Delta})$. In this way, we see that Jacobians of graphs are easily computable. Indeed, if one puts the reduced Laplacian in Smith normal form, one obtains a decomposition of $\mathrm{Jac}(G)$ as a direct sum of cyclic groups.

Example 2.3. In the previous lecture, we computed the Picard group of the graph G depicted in Figure 1. We found that $\mathrm{Pic}(G) \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. The reduced Laplacian

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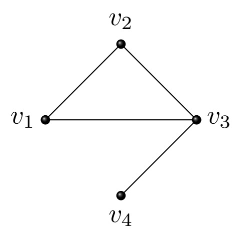

FIGURE 1. A simple graph.

obtained by removing the third row and column is

$$\widetilde{\Delta} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The determinant of $\widetilde{\Delta}$ is -3, so $\mathrm{Jac}(G) \cong \mathbb{Z}/3\mathbb{Z}$.

Recall that a *spanning tree* in a graph G is a subgraph that contains every vertex and is a tree. Perhaps the most well-known result concerning reduced graph Laplacians is Kirchoff's Matrix Tree Theorem.

Matrix Tree Theorem. The absolute value of the determinant of the reduced graph Laplacian of a graph G is equal to the number of spanning trees in G.

To prove the Matrix Tree Theorem, we choose an orientation of the graph G, and let E be the matrix with columns indexed by the edges of G and rows indexed by the vertices of G, given by

$$E_{ve} = \begin{cases} 0 & \text{if } e \text{ is not incident to } v \\ 1 & \text{if } v \text{ is the head of } e \\ -1 & \text{if } v \text{ is the tail of } e. \end{cases}$$

Lemma 2.4. *Let G be a graph. Then $\Delta(G) = -EE^T$.*

Proof. If $i \neq j$, then the (i, j)th entry of EE^T is given by multiplying the row of E corresponding to vertex i by the column of E^T corresponding to vertex j. We see that the nonzero entries of each vector correspond to edges incident to the given vertex, and the two vectors share a nonzero entry when there is an edge incident to both vertex i and vertex j. For each such edge, one of the vectors contains a 1 and the other contains a -1. Thus, the (i, j)th entry of EE^T is the negative of the number of edges incident to both vertex i and vertex j.

The (i, i)th entry of EE^T is given by multiplying the row of E corresponding to vertex i by its own transpose. Again, the nonzero entries of this vector correspond to edges incident to vertex i. Thus, the (i, i)th entry is the number of edges incident to vertex i. □

Proof of the Matrix Tree Theorem. Let $\widetilde{\Delta}$ denote the reduced graph Laplacian obtained by removing the row and column corresponding to some vertex v from the graph Laplacian Δ. Let $\widetilde{E}$ denote the minor of E obtained by removing the row of E corresponding to the same vertex v. By the Cauchy-Binet formula for the determinant, we have

$$\det \widetilde{\Delta} = \sum_S \det \widetilde{E}_S \det \widetilde{E}_S^T = \sum_S \det \widetilde{E}_S^2,$$

where the sum is over all subsets of the edges of size $|V(G)| - 1$. Now, if no edge of $S \subset E(G)$ is incident to the vertex $w \neq v$, then $\widetilde{E}_S$ contains a row of all zeros, and therefore has determinant zero. If no edge of S is incident to the vertex v, then the sum of the rows of $\widetilde{E}_S$ is the zero vector, hence $\widetilde{E}_S$ has determinant zero. We therefore see that, if S is not a spanning tree, then $\det \widetilde{E}_S = 0$.

On the other hand, if S is a spanning tree, we will show that $\det \widetilde{E}_S = \pm 1$. For every vertex $w \in V(G)$, consider the unique path in S from w to v. Adding the columns of $\widetilde{E}_S$ corresponding to the edges in this path, we obtain a vector with ± 1 in the row corresponding to w, and a 0 in every other entry. By performing these elementary column operations to every column of $\widetilde{E}_S$, we obtain a matrix in which every row and column has precisely one nonzero entry, and this nonzero entry is ± 1. The determinant of such a matrix is ± 1. □

Corollary 2.5. *For any graph G, the order of* $\mathrm{Jac}(G)$ *is equal to the number of spanning trees in G. In particular,* $\mathrm{Jac}(G)$ *is a finite abelian group.*