CHIP FIRING

3. Effective and Reduced Divisors

The main topic for today is the theory of v-reduced divisors, which are canonical representatives of divisor classes on graphs, depending only on the choice of a base vertex v. We will prove the existence and uniqueness of v-reduced divisors, along with some of the fundamental properties that make them essential tools in this subject.

Definition 3.1. *A divisor $D = \sum_{v \in V(G)} D(v)v$ is* effective *if $D(v) \geq 0$ for all $v \in V(G)$.*

Not every divisor on a graph is equivalent to an effective divisor. For example, a divisor of negative degree cannot be effective. A divisor of degree 0 is effective if and only if it is the zero divisor. To determine whether a given divisor is equivalent to an effective divisor, we use the theory of v-reduced divisors. These are canonical representatives of divisor classes in $\mathrm{Pic}(G)$, depending only on the choice of a base vertex v.

Definition 3.2. *Let G be a graph and let $v \in V(G)$. A divisor $D = \sum_{w \in V(G)} D(w)w$ is* effective away from v *if $D(w) \geq 0$ for all $w \neq v$.*

Lemma 3.3. *Every divisor class contains a representative that is effective away from v.*

Proof. Let D be a divisor on a graph G. Recall that the group $\mathrm{Jac}(G)$ is finite, and let $m = |\mathrm{Jac}(G)|$. For every vertex w, $m[w - v]$ is trivial. For each vertex w, choose an integer α_w so that $\alpha_w m + D(w) \geq 0$. Then the divisor

$$D + m \sum_{w \in V(G) \smallsetminus \{v\}} \alpha_w [w - v]$$

is equivalent to D and effective away from v. □

We introduce a couple more pieces of notation, which will be useful for describing explicit equivalences of divisors. Let $\vec{f}$ be an element of $\mathbb{Z}^{V(G)}$, considered as the domain of Δ. We write $D_{\vec{f}}$ for $\Delta \vec{f} \in \mathrm{Div}(G)$. Also, for $A \subset V(G)$, we let $D_A = D_{\chi(A)}$. In other words, adding D_A to a divisor is the same as "firing all vertices of A at once". The following is the key definition.

Definition 3.4. *A divisor D is* v-reduced *if it is effective away from v, and for any subset $A \subset V(G) \backslash \{v\}$, $D + D_A$ is not effective away from v.*

The primary goal of this lecture is to prove that every divisor is equivalent to a unique v-reduced divisor.

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Proposition 3.5. *Every divisor class contains a unique v-reduced divisor.*

Proof. (Existence) Order the vertices so that every vertex other than v has a neighbor that comes after it. (To do this, choose a spanning tree of G and an orientation of this tree with v as the unique sink.)

Given a divisor class in $\mathrm{Pic}(G)$, by Lemma 3.3 there exists a representative D that is effective away from v. Either D is v-reduced or there is a subset $A \subset V(G) \setminus \{v\}$ such that $D+D_A$ is effective away from v. Then, because A is not the whole graph, it has some neighbor that comes after it, so $D+D_A$ is larger than D in the lexicographic ordering. Now replace D by $D+D_A$ and repeat. This procedure terminates because there are only finitely many divisors of a given degree that are effective away from v and that are larger than a given divisor in the lexicographic ordering.

(Uniqueness) Suppose $D \sim D'$ are distinct divisors, both effective away from v. We show that only one of D, D' can be v-reduced. Because D and D' are equivalent, there exists a vector $\vec{f}$ with $D' = D + D_{\vec{f}}$, and because $D \neq D'$, $\vec{f}$ is non-constant. By interchanging D and D' if necessary, we may assume that $\vec{f}$ achieves its maximum on a set $A \subset V(G) \setminus \{v\}$. We will show that $D' + D_A$ is effective away from v, and thus D' is not v-reduced. For any vertex $w \notin A$, the coefficient of w in $D' + D_A$ is greater than or equal to $D(w)$. Thus, if $w \neq v$, this coefficient is nonnegative. For any vertex $w \in A$, the coefficient of w in $D' + D_A$ is

$$D'(w) - \mathrm{outdeg}_w(A) \geq D'(w) - \sum_{uw \in E(G)} [\vec{f}(w) - \vec{f}(u)] = D(w) \geq 0.$$

This completes the proof. □