CHIP FIRING

6. Orientable Divisors, continued

In this lecture, we continue our discussion of orientable divisors. In the previous lecture, we saw that every divisor of degree at least g on a graph of genus g is equivalent to an effective divisor. We will now see that this bound is sharp. On any graph of genus g, there exist divisors of degree $g-1$ that are not equivalent to an effective divisor.

Lemma 6.1. *Let $\mathcal{O}$ be an acyclic orientation of a graph G. Then $D_{\mathcal{O}}$ is not equivalent to an effective divisor.*

Proof. Given a vector $\vec{f}$, we will show that the divisor $D_{\mathcal{O}} + D_{\vec{f}}$ is not effective. Let $A \subseteq V(G)$ be the set of vertices where $\vec{f}$ achieves its maximum. Because $\mathcal{O}$ is acyclic, there exists a vertex $v \in A$ that is initial in A. In other words, every edge adjacent to v whose other endpoint is also in A is oriented away from v. It follows that $\text{indeg}_{\mathcal{O}}(v) \leq \text{outdeg}_A(v)$. We therefore see that

$$D_{\mathcal{O}}(v) + D_{\vec{f}}(v) \leq \text{outdeg}_A(v) - 1 - \text{outdeg}_A(v) = -1,$$

so $D_{\mathcal{O}} + D_{\vec{f}}$ is not effective. □

Corollary 6.2. *Let G be a graph of genus g. There exists a divisor of degree $g-1$ on G that is not equivalent to an effective divisor.*

Proof. By Lemma 6.1, it suffices to construct an acyclic orientation of G. To do this, choose a total ordering on the vertices of G, and orient each edge of G so that the tail precedes the head in the total ordering. The resulting orientation is acyclic, because the first vertex in any directed path precedes that path's final vertex in the total ordering. □

In fact, acyclic orientations can be used to distinguish between divisors that are equivalent to effective divisors and those that are not.

Lemma 6.3. *For any divisor D on a graph G, either D is equivalent to an effective divisor or there is an acyclic orientation $\mathcal{O}$ such that $D_{\mathcal{O}} - D$ is equivalent to an effective divisor (but not both).*

Proof. We may assume that D is v-reduced for some vertex v. We construct an orientation $\mathcal{O}$ of G by running Dhar's Burning Algorithm, and orienting each edge in the direction it burns. To see that $\mathcal{O}$ is acyclic, note that every time a vertex burns, all of the adjacent edges burn, hence one cannot proceed via a directed path from a newly burnt to a previously burnt vertex.

Date: January 25, 2019.

For all $w \neq v$, because the vertex w burns, we have $D(w) \leq D_{\mathcal{O}}(w)$. If $D(v) \geq 0$, then D is effective. Otherwise, $D(v) \leq -1 = \text{indeg}_{\mathcal{O}} v - 1 = D_{\mathcal{O}}(v)$, so $D \leq D_{\mathcal{O}}$ everywhere. In other words, $D_{\mathcal{O}} - D$ is effective. □

Another fact that we saw in the previous lecture is that every divisor of degree $g-1$ on a graph of genus g is equivalent to a v-connected orientable divisor. We will now show that this v-connected orientable divisor is unique.

Proposition 6.4. *Let $\mathcal{O}$ and $\mathcal{O}'$ be orientations of a graph G, and let v be a vertex of G. If $D_{\mathcal{O}}$ and $D_{\mathcal{O}'}$ are equivalent but not equal, then at most one of them is v-connected.*

Proof. If $D_{\mathcal{O}}$ and $D_{\mathcal{O}'}$ are equivalent, then there exists a vector $\vec{f}$ such that $D_{\mathcal{O}'} = D_{\mathcal{O}} + D_{\vec{f}}$. Let $A \subset V(G)$ be the set of vertices where $\vec{f}$ achieves it maximum. If $D_{\mathcal{O}}$ and $D_{\mathcal{O}'}$ are not equal, then $A \neq V(G)$. Note that

$$\deg(D_{\mathcal{O}'}|_A) \geq |E(G[A])| - |A|,$$

with equality if and only if all edges in the cut (A, A^c) are directed away from A in the orientation $\mathcal{O}'$. Similarly,

$$\deg(D_{\mathcal{O}}|_{A^c}) \geq |E(G[A^c])| - |A^c|,$$

with equality if and only if all edges in the cut (A, A^c) are directed toward A in the orientation $\mathcal{O}$.

By our choice of the set A, we have

$$\deg(D_{\mathcal{O}'}|_A) \leq \deg(D_{\mathcal{O}}|_A) - |E(A, A^c)|.$$

Putting all these inequalities together, we see that

$$\begin{aligned}
&|E(G[A])| - |A| \\
&\leq \deg(D_{\mathcal{O}'}|_A) \\
&\leq \deg(D_{\mathcal{O}}|_A) - |E(A, A^c)| \\
&= |E(G)| - |V(G)| - \deg(D_{\mathcal{O}}|_{A^c}) - |E(A, A^c)| \\
&\leq |E(G)| - |V(G)| + |A^c| - |E(G[A^c])| - |E(A, A^c)| \\
&= |E(G[A])| - |A|.
\end{aligned}$$

Because the expression above is bounded above and below by the same quantity $|E(G[A])| - |A|$, we see that equality must hold in each of the lines above. It follows that all edges in the cut (A, A^c) are directed away from A in the orientation $\mathcal{O}'$, and toward A in the orientation $\mathcal{O}$. Therefore, if $v \in A$, then $\mathcal{O}$ cannot be v-connected, and if $v \notin A$, then $\mathcal{O}'$ cannot be v-connected. □