CHIP FIRING

7. Break Divisors

In previous lectures, we have studied canonical representatives of divisor classes on graphs. More specifically, given a vertex v on a graph of genus g, every divisor class contains a unique v-reduced representative, and every divisor class of degree $g-1$ contains a unique v-connected orientable representative. A drawback of these theories is that these representatives depend on the choice of vertex v. In this lecture, we discuss canonical representatives of degree g divisors that do not depend on any choices.

Definition 7.1. *Let G be a graph, and let T be a spanning tree of G. For each edge e not in T, let v_e be one of its endpoints. A divisor of the form*

$$D = \sum_{e \notin T} v_e$$

is called a break divisor.

By definition, a break divisor is effective of degree g.

Example 7.2. The graph pictured in Figure 1 has 8 spanning trees. There are 2 edges in the complement of each spanning tree, and for each of the 2 edges, there are 2 choices of endpoints. We note, however, that 2 distinct choices of spanning tress and endpoints do not necessarily yield distinct break divisors. There are in fact 8 break divisors on the graph, all of which are pictured in Figure 1. The reader is encouraged to check that all of these are break divisors, and there are no others.

There is a direct connection between break divisors and orientable divisors.

Proposition 7.3. *Let G be a graph and v a vertex of G. A divisor D on G is a break divisor if and only if $D - v$ is a v-connected orientable divisor.*

Proof. First, let $\mathcal{O}$ be a v-connected orientation of G. Since $\mathcal{O}$ is v-connected, there exists a spanning tree T with unique source v. For $w \neq v$, there is exactly one edge in T that points toward w. Thus, if $\alpha(w)$ is the number of edges not in T that are directed toward w, we have

$$D_{\mathcal{O}}(w) = \mathrm{indeg}_{\mathcal{O}}(w) - 1 = 1 + \alpha(w) - 1 = \alpha(w).$$

Similarly, since v is a source in T, we have

$$D_{\mathcal{O}}(v) = \alpha(v) - 1.$$

It follows that $D_{\mathcal{O}} + v$ is a sum, over edges not in the spanning tree T, of one endpoint for each edge. In other words, $D_{\mathcal{O}} + v$ is a break divisor.

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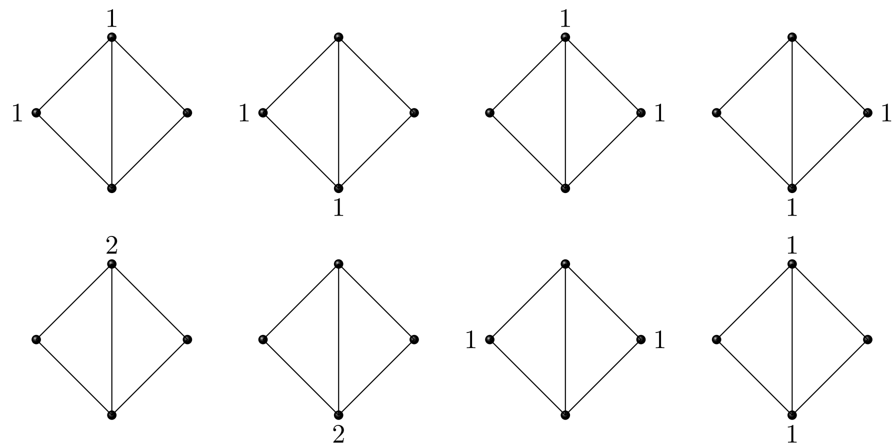

FIGURE 1. Break divisors on a graph of genus 2

To see the reverse direction, let D be a break divisor. By definition, there exists a spanning tree T and, for each edge $e \notin T$, an endpoint v_e such that $D = \sum_{e \notin T} v_e$. We construct an orientation $\mathcal{O}$ by first orienting the tree so that v is the unique source, and then orienting each edge $e \notin T$ so that it points toward v_e. As before, if $w \neq v$, then there is exactly one edge in T that points toward w. Thus,

$$D_{\mathcal{O}}(w) = \mathrm{indeg}_{\mathcal{O}}(w) - 1 = |e \notin T \text{ such that } w = v_e|.$$

Similarly, since v is a source in T, we have

$$D_{\mathcal{O}}(v) = |e \notin T \text{ such that } v = v_e| - 1.$$

It follows that $D_{\mathcal{O}} + v = \sum_{e \notin T} v_e = D$. □

Corollary 7.4. *Let G be a graph of genus g. Every divisor class of degree g on G contains a unique break divisor representative.*

Proof. Let D be a divisor on G of degree g, and let v be a vertex of G. Then $D - v$ has degree $g - 1$, so $D - v$ is equivalent to a v-connected orientable divisor $D_{\mathcal{O}}$. By Proposition 7.3, we see that $D_{\mathcal{O}} + v$ is a break divisor equivalent to D.

The uniqueness of this break divisor follows from the uniqueness of $D_{\mathcal{O}}$. Specifically, if $D \sim E$ are break divisors, then by Proposition 7.3, $D - v \sim E - v$ are v-connected orientable divisors. It follows that $D - v = E - v$, so $D = E$. □

As a consequence, we return to a fact that we have already seen.

Corollary 7.5. *Let G be a graph of genus g. Every divisor of degree at least g on G is equivalent to an effective divisor.*

Proof. Let D be a divisor on G of degree $d \geq g$, and let v be a vertex of G. Then $D - (d - g)v$ has degree g, so it is equivalent to a break divisor E. By definition, E is effective, so $E + (d - g)v$ is an effective divisor equivalent to D. □