CHIP FIRING

8. The Rank of a Divisor

A key invariant of a divisor is its *rank*. It is this invariant that powers the connection between chip firing and algebraic geometry.

Definition 8.1. *Let D be a divisor on a graph. If D is not equivalent to an effective divisor, we say that D has* rank -1*. Otherwise, we define the* rank *of D to be the largest integer r such that, for all effective divisors E of degree r, $D-E$ is equivalent to an effective divisor.*

Example 8.2. Since the only effective divisor of degree zero is the identically zero divisor, we see that a divisor has nonnegative rank if and only if it is equivalent to an effective divisor.

Example 8.3. Computing the rank of a divisor can be thought of as a game, in which our opponent is allowed to "steal" r chips from wherever they like, and our task is to perform a sequence of chip firing moves that eliminates the debt created by our opponent. If we can win this game regardless of which r chips our opponent chooses to steal, then the divisor has rank at least r.

Consider, for example, the two divisors of degree 2 depicted in Figure 1. Both divisors are effective, so they both have nonnegative rank. To determine whether a divisor D has rank at least 1, we check to see if $D - v$ is equivalent to an effective divisor for each vertex v. For the divisor on the left, if v is the bottom right vertex, then Dhar's Burning Algorithm shows that $D - v$ is v-reduced, but not effective. It follows that $D-v$ is not equivalent to an effective divisor, hence D has rank less than 1. Since we have already seen that D has nonnegative rank, we see that it must have rank 0.

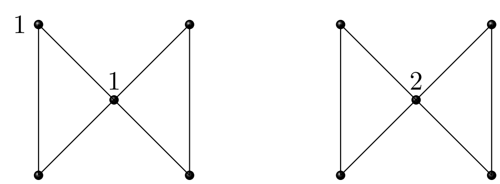

Figure 1. Two divisors of the same degree and different rank on a graph of genus 2

On the other hand, the divisor K on the right has rank at least 1. Again, to see this we must check whether $K - v$ is equivalent to an effective divisor for each vertex v. By symmetry, it suffices to consider the case where v is the center vertex, and

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the case where v is the bottom right vertex. If v is the center vertex, then $K - v$ is effective. If v is the bottom right vertex, then we use Dhar's Burning Algorithm to compute the v-reduced divisor equivalent to $K - v$. This divisor is obtained by firing the 3 vertices in the lefthand triangle, and the result is effective. Therefore, the divisor K has rank at least 1.

To see that the rank of K is exactly 1, note that D is an effective divisor of degree 2, and K is not equivalent to D. Therefore, $K - D$ is not equivalent to 0. Since the zero divisor is the only effective divisor of degree 0, we see that $K - D$ is not equivalent to an effective divisor. It follows that the rank of K is less than 2.

We record a few other simple observations about ranks of divisors.

Lemma 8.4. *Let D be a divisor on a graph of genus g. Then* $\mathrm{rk}(D) \geq \deg(D) - g$*.*

Proof. If $\deg(D) < g$, the result is obvious, since every divisor has rank at least -1. Otherwise, let E be an effective divisor of degree $\deg(D) - g$. Then $\deg(D - E) = g$. We have seen that every divisor of degree g is equivalent to an effective divisor, so $D - E$ is equivalent to an effective divisor. Since E was arbitrary, we see that the D has rank at least $\deg(D) - g$. □

Lemma 8.5. *Let D_1, D_2 be divisors of nonnegative rank on a graph G. Then*

$$\mathrm{rk}(D_1 + D_2) \geq \mathrm{rk}(D_1) + \mathrm{rk}(D_2).$$

Proof. Let E be an effective divisor of degree $\mathrm{rk}(D_1) + \mathrm{rk}(D_2)$. Write $E = E_1 + E_2$, where E_1 and E_2 are both effective and $\deg(E_i) = \mathrm{rk}(D_i)$. By definition, $D_i - E_i$ is equivalent to an effective divisor F_i. It follows that $(D_1 + D_2) - E = (D_1 - E_1) + (D_2 - E_2)$ is equivalent to $F_1 + F_2$, an effective divisor. Since E was arbitrary, we see that $D_1 + D_2$ has rank at least $\mathrm{rk}(D_1) + \mathrm{rk}(D_2)$. □

We make the following definition.

Definition 8.6. *Let D be a divisor on a graph G. We define*

$$\deg^+(D) = \sum_{v \in V(G), D(v) > 0} D(v).$$

We can now give an alternate characterization of the rank.

Proposition 8.7. *Let D be a divisor on a graph G. Then*

$$\mathrm{rk}(D) = \min_{\substack{D' \sim D \\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}})\} - 1.$$

Proof. We first show that

$$\mathrm{rk}(D) < \min_{\substack{D' \sim D \\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}})\}.$$

To see this, let $(D', \mathcal{O})$ be a pair that achieves the minimum, let

$$E^+ = \sum_{v \in V(G), D'(v) > D_{\mathcal{O}}(v)} (D'(v) - D_{\mathcal{O}}(v))v,$$

and $E^- = E^+ - D' + D_{\mathcal{O}}$. Note that both E^+ and E^- are effective, and

$$\deg(E^+) = \deg^+(D' - D_{\mathcal{O}}) = \min_{\substack{D' \sim D \\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}})\}.$$

Rearranging the terms, we see that $D - E^+ \sim D' - E^+ = D_{\mathcal{O}} - E^-$. Since $\mathcal{O}$ is acyclic, the divisor $D_{\mathcal{O}} - E^-$ is not equivalent to an effective divisor. Therefore, the rank of D is smaller than $\deg(E^+)$.

We now show that

$$\operatorname{rk}(D) \geq \min_{\substack{D' \sim D \\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}})\} - 1.$$

To see this, let E be an effective divisor of degree

$$\deg(E) = \min_{\substack{D' \sim D \\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}})\} - 1.$$

If $D - E$ is not equivalent to an effective divisor, then there exists an acyclic orientation $\mathcal{O}$ such that $D_{\mathcal{O}} - (D - E)$ is equivalent to an effective divisor E'. In other words, there exists a divisor $D' \sim D$ such that $D_{\mathcal{O}} - D' + E = E'$. Rearranging, we see that $D' - D_{\mathcal{O}} = E - E'$. Therefore,

$$\deg^+(D' - D_{\mathcal{O}}) = \deg^+(E - E') \leq \deg(E) = \min_{\substack{D' \sim D \\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}})\} - 1,$$

a contradiction. It follows that $D - E$ is equivalent to an effective divisor, and the result follows. $\square$