Sandwich Theorem

One More Limit Law:

Suppose that $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ exists.

If $f(x) \le g(x)$ near a, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

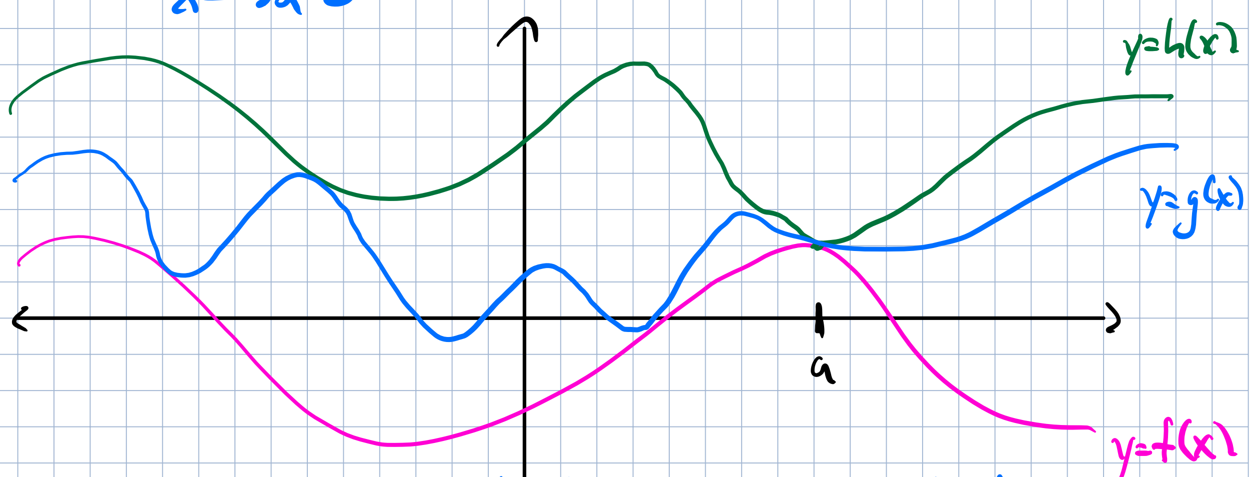
Sandwich Theorem (Squeeze Theorem):

Let $f(x)$, $g(x)$, and $h(x)$ be functions.

Suppose that $f(x) \le g(x) \le h(x)$ near a

and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$.

Then $\lim_{x\to a} g(x) = L$.



Since $f(x)$ and $h(x)$ have the same limit at a, and $g(x)$ is sandwiched between them, $g(x)$ also has the same limit at a.

Ex: Compute $\lim_{x \to 0} x^2 \cdot \sin\left(\frac{\pi}{x}\right)$.

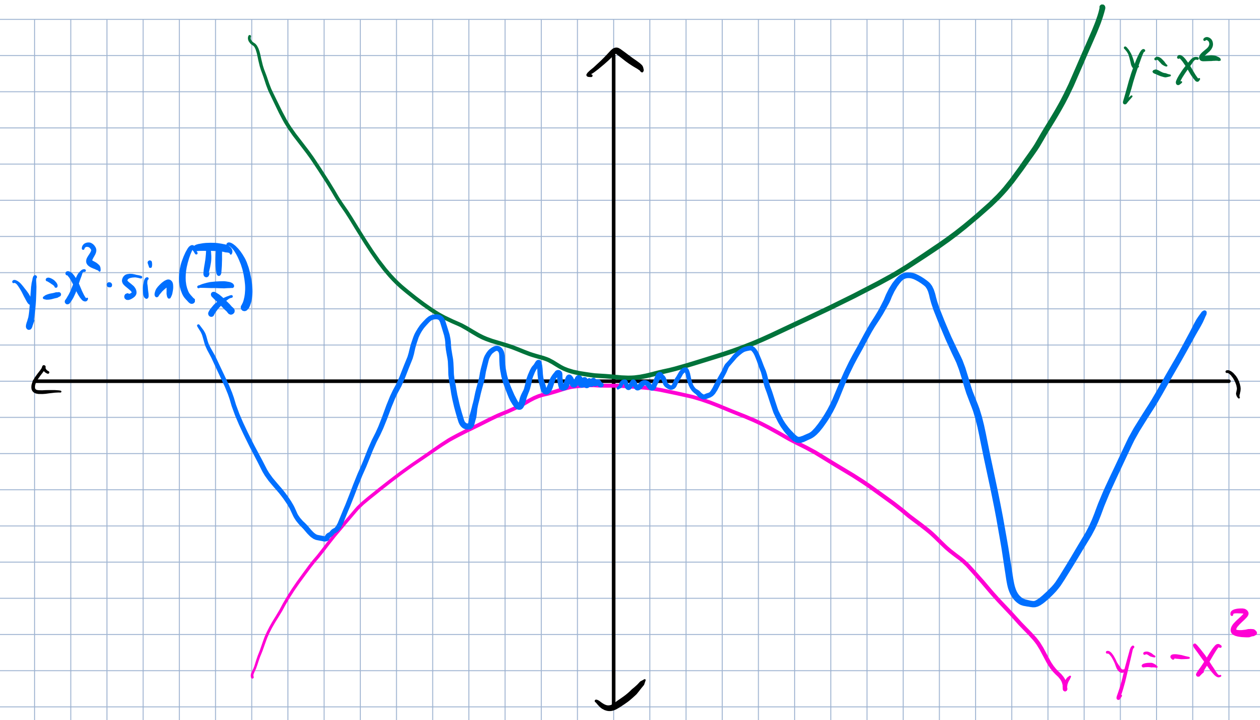
$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

$x^2 \geq 0$ for all x, so

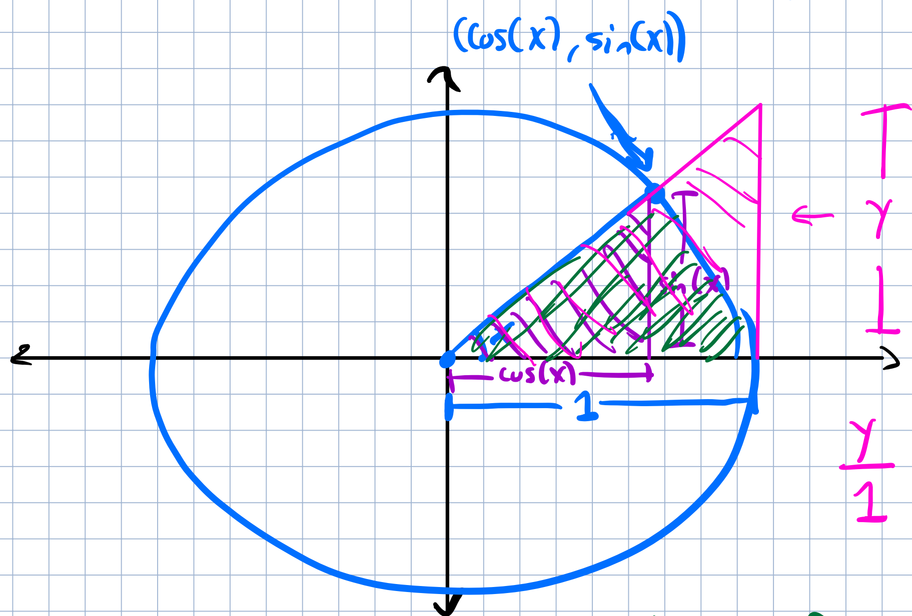
$$-x^2 \leq x^2 \cdot \sin\left(\frac{\pi}{x}\right) \leq x^2$$

x^2 is a polynomial, so it is continuous.
Therefore $\lim_{x \to 0} x^2 = 0^2 = 0$.
Similarly, $\lim_{x \to 0} -x^2 = -0^2 = 0$.

By the Sandwich Theorem, we see that
$$\lim_{x \to 0} x^2 \cdot \sin\left(\frac{\pi}{x}\right) = 0.$$



Application: Compute $\lim_{x\to 0} \frac{\sin(x)}{x}$.



$$\frac{y}{1} = \frac{\sin(x)}{\cos(x)}$$

Area of purple triangle ≤ Area of green sector ≤ Area of pink triangle

$$\cancel{\tfrac{1}{2}}\sin(x)\cdot\cos(x) \le \cancel{\tfrac{1}{2}}x \le \cancel{\tfrac{1}{2}}\cdot 1\cdot\frac{\sin(x)}{\cos(x)}$$

$$\cos(x) \le \frac{x}{\sin(x)} \le \frac{1}{\cos(x)}$$

$$\frac{1}{\cos(x)} \ge \frac{\sin(x)}{x} \ge \cos(x)$$

Since $\cos(x)$ is continuous,

$$\lim_{x \to 0} \cos(x) = \cos(0) = 1$$

Similarly, since $\frac{1}{\cos(x)}$ is continuous,

$$\lim_{x \to 0} \frac{1}{\cos(x)} = \frac{1}{\cos(0)} = \frac{1}{1} = 1.$$

By the Sandwich Theorem,

$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1.}$$

Ex: Compute $\lim\limits_{x \to 0} \frac{1-\cos(x)}{x}$

$$\lim_{x \to 0} \frac{1-\cos(x)}{x} \cdot \frac{1+\cos(x)}{1+\cos(x)}$$

$$= \lim_{x \to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))} = \lim_{x \to 0} \frac{\sin^2(x)}{x(1+\cos(x))}$$

$$= \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \lim_{x \to 0} \frac{\sin(x)}{1+\cos(x)}$$

$$= 1 \cdot \frac{\sin(0)}{1+\cos(0)} = 1 \cdot \frac{0}{1+1}$$

$$= 0.$$

$$\boxed{\lim_{x \to 0} \frac{1-\cos(x)}{x} = 0.}$$

Ex: Compute $\lim_{x \to 0} 4 + |x| \cdot \cos\left(\frac{2}{x}\right)$.

$$-1 \le \cos\left(\frac{2}{x}\right) \le 1$$

$$-|x| \le |x| \cdot \cos\left(\frac{2}{x}\right) \le |x|$$

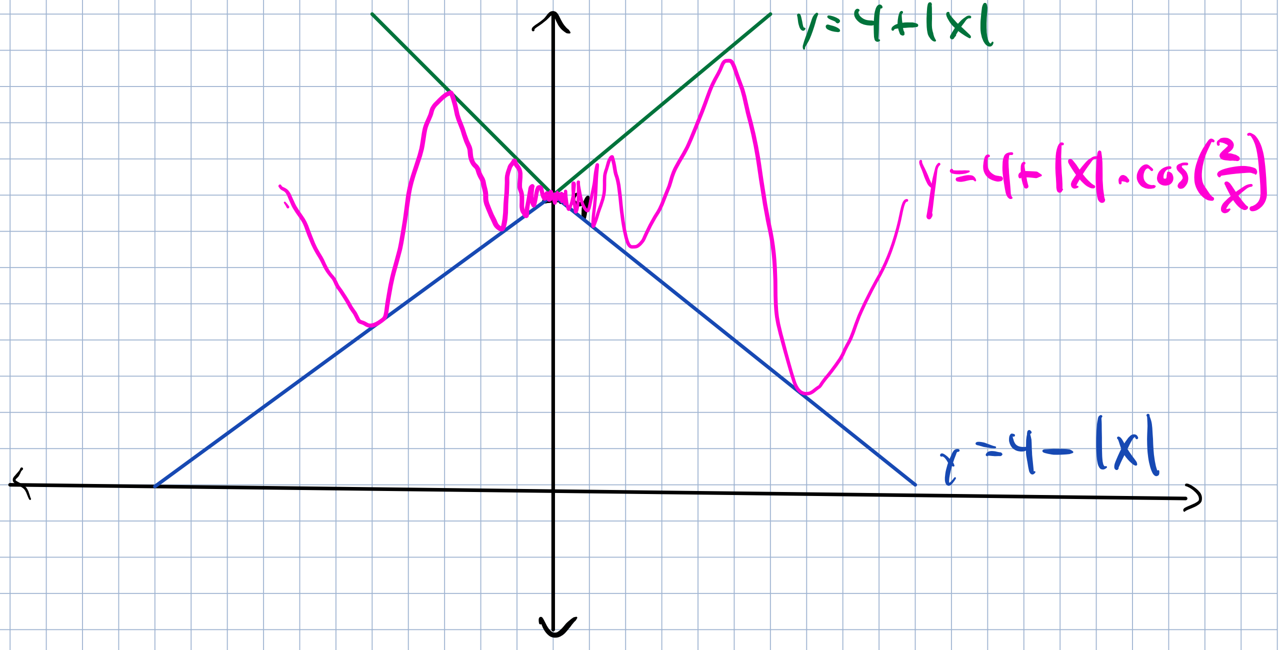
$$4 - |x| \le 4 + |x| \cdot \cos\left(\frac{2}{x}\right) \le 4 + |x|.$$

$$\lim_{x \to 0} 4 - |x| = 4 - |0| = 4$$

$$\lim_{x \to 0} 4 + |x| = 4 + |0| = 4.$$

By the Sandwich Theorem,

$$\lim_{x \to 0} 4 + |x| \cdot \cos\left(\frac{2}{x}\right) = 4.$$



$y = 4 + |x|$
$y = 4 + |x| \cdot \cos\left(\frac{2}{x}\right)$
$y = 4 - |x|$