Product Rule

$$\boxed{\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)}$$

Why?

$$\frac{d}{dx}\left[f(x)g(x)\right] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} g(x+h)\frac{f(x+h)-f(x)}{h} + f(x)\frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} g(x+h) \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= g(x)f'(x) + f(x)\cdot g'(x)$$

Ex: Compute $\frac{d}{dx}\left[\underbrace{(x^7+3x^5-2)}_{f}\underbrace{(x^9+12x^2+3x)}_{g}\right]$

$f(x) = x^7+3x^5-2$ $\quad$ $g(x) = x^9+12x^2+3x$

$f'(x) = 7x^6+3\cdot 5x^4$ $\quad$ $g'(x) = 9x^8+12\cdot 2x+3$

$= 7x^6 + 15x^4$ $\qquad = 9x^8 + 24x + 3$

PRODUCT RULE

$$\frac{d}{dx}[f \cdot g] = f \cdot g' + g \cdot f'$$

$$= (x^7 + 3x^5 - 2)(9x^8 + 24x + 3)$$
$$+ (x^9 + 12x^2 + 3x)(7x^6 + 15x^4)$$

Ex: $\frac{d}{dx}\left[(x^{10} + x^5) \cdot (3x^3 + 4x^2)\right]$

PRODUCT RULE
$$= (x^{10} + x^5) \cdot \frac{d}{dx}[3x^3 + 4x^2] + (3x^3 + 4x^2) \cdot \frac{d}{dx}[x^{10} + x^5]$$

$$= (x^{10} + x^5) \cdot (3 \cdot 3x^2 + 4 \cdot 2x) + (3x^3 + 4x^2) \cdot (10x^9 + 5x^4)$$

$$= (x^{10} + x^5)(9x^2 + 8x) + (3x^3 + 4x^2)(10x^9 + 5x^4)$$

Ex: Compute $\frac{d}{dx}[\sqrt{x}]$

Note: $(\sqrt{x} = x^{1/2})$

$$x = \sqrt{x} \cdot \sqrt{x}$$

$$1 = \frac{d}{dx}[x] = \frac{d}{dx}[\sqrt{x} \cdot \sqrt{x}]$$

PRODUCT RULE
$$= \sqrt{x} \cdot \frac{d}{dx}[\sqrt{x}] + \sqrt{x} \cdot \frac{d}{dx}[\sqrt{x}]$$

$$= 2\sqrt{x} \cdot \frac{d}{dx}[\sqrt{x}]$$

$$\frac{1}{2\sqrt{x}} = \frac{d}{dx}[\sqrt{x}]$$

$$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{1}{2\sqrt{x}}$$

OR

$$\boxed{\frac{d}{dx}\left[x^{1/2}\right] = \frac{1}{2}\cdot x^{-1/2}}$$

Quotient Rule

Find $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$.

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g(x)^2}}$$

Why?

$$f'(x) = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\cdot g(x)\right]$$

PRODUCT RULE

$$= \frac{f(x)}{g(x)}\cdot g'(x) + g(x)\cdot\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$$

$$f'(x) - \frac{f(x)g'(x)}{g(x)} = g(x)\cdot\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$$

$$\frac{g(x)f'(x) - f(x)g'(x)}{g(x)} = g(x)\,\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$$

$$\frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2} = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$$

Ex: Compute $\frac{d}{dx}\left[\frac{x^2+1}{3x}\right]$.

$f(x) = x^2+1 \qquad g(x) = 3x$

$f'(x) = 2x \qquad g'(x) = 3$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \overset{\text{QUOTIENT RULE}}{=} \frac{g(x)f'(x) - f(x)\cdot g'(x)}{g(x)^2}$$

$$= \frac{(3x)\cdot(2x) - (x^2+1)\cdot(3)}{(3x)^2}$$

$$= \frac{6x^2 - 3x^2 - 3}{9x^2} = \frac{3x^2-3}{9x^2}$$

$$= \frac{x^2-1}{3x^2}$$

Ex: Compute $\frac{d}{dx}\left[\frac{3x^3-5x^2}{x^2-2}\right]$.

$$f(x) = 3x^3 - 5x^2 \qquad g(x) = x^2 - 2$$

$$f'(x) = 9x^2 - 10x \qquad g'(x) = 2x$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g(x)^2}$$

$$= \frac{(x^2-2)(9x^2-10x) - (3x^3-5x^2)(2x)}{(x^2-2)^2}$$

$$= \frac{9x^4 \cancel{-10x^3} - 18x^2 + 20x - 6x^4 \cancel{+10x^3}}{(x^2-2)^2}$$

$$= \frac{3x^4 - 18x^2 + 20x}{(x^2-2)^2}$$

Last time we saw $\frac{d}{dx}[x^n] = nx^{n-1}$ when n is a positive whole number.

Today we saw $\frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{-1/2}$

Now, let's compute $\frac{d}{dx}[x^n]$ when $\cancel{n \le -1}$ n is a negative whole number.

$$\frac{d}{dx}\left[x^n\right] = \frac{d}{dx}\left[\frac{1}{x^{-n}}\right]$$

$f(x) = 1$ $\qquad$ $g(x) = x^{-n}$

$f'(x) = 0$ $\qquad$ $g'(x) = -n x^{-n-1}$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g(x)^2}$$

$$= \frac{x^{-n}\cdot 0 - 1\cdot(-nx^{-n-1})}{(x^{-n})^2}$$

$$= \frac{nx^{-n-1}}{x^{-2n}} = nx^{-n-1}\,x^{2n}$$

$$= nx^{(-n-1+2n)} = nx^{n-1}$$

POWER RULE holds for negative numbers as well

Ex: Compute $\frac{d}{dx}\left[3x^5 - \frac{7}{x^3}\right]$

$$= \frac{d}{dx}\left[3x^5 - 7x^{-3}\right]$$

$$= 3 \cdot 5x^4 - 7(-3x^{-4})$$

$$= 15x^4 + 21x^{-4} = 15x^4 + \frac{21}{x^4}$$